Chapter 1

The Wonderful World of Differential Equations

> The essential fact is simply that all the pictures which science now draws of nature, and which alone seem capable of according with observational fact, are mathematical pictures. . . .
>
> Sir James Jeans[1]

1.1 What are Differential Equations?

Just what are differential equations? Following the wisdom of the old Chinese proverb that "one picture is worth more than a thousand words," we defer our answer until we have provided a picture of sorts. Table 1.1 is, so to speak, a collage of various types of differential equations. With one exception, these are well-known equations drawn from different scientific and technical disciplines. A sense of their importance may be realized from their ability to mathematically describe, or model, real-life situations. The equations come from the diverse disciplines of demography, ecology, chemical kinetics, architecture, physics, mechanical engineering, quantum mechanics, electrical engineering, civil engineering, meteorology, and a relatively new science called *chaos*. The same differential equation may be important to several disciplines, although for different reasons. For example, demographers, ecologists, and mathematical biologists would immediately recognize

$$\frac{dp}{dt} = rp,$$

the first equation in Table 1.1, as the *Malthusian law of population growth*. It is used to predict populations of certain kinds of organisms reproducing under ideal conditions—

[1]See [31, ch. 5]. Sir James Jeans (1877–1946) was a British mathematical physicist, Cambridge University lecturer, Princeton University professor of applied mathematics, and author of a number of popular works of science, of which *The Mysterious Universe* [31] was one of his most famous. His treatise, *Problems of Cosmogony and Stellar Dynamics* (1917), on the behavior of fluids in space contributed to a greater understanding of the origin and evolution of the universe.

whereas physicists, chemists, and nuclear engineers would be more inclined to regard the equation as a mathematical portrayal, or model, of radioactive decay. Even many economists and mathematically minded investors would recognize this differential equation, but in a totally different context: it also models future balances of investments earning interest at rates compounded continuously.

Another example is the *van der Pol equation*

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0,$$

which came from modeling oscillations of currents in the nonlinear electrical circuits of the first commercial radios. For many years, it was the subject of research by electrical engineers and mathematicians alike.

Table 1.1: **Differential Equations Modeling Real-Life Situations**

Differential Equation	**Situation**
$\frac{dp}{dt} = rp$	The *Malthusian law of population growth* is used to model the populations of certain kinds of organisms living in ideal environments for limited lengths of time t. It gives the rate at which a population p changes with respect to t. The value of the constant r depends on the organism.
$\frac{dx}{dt} = k(A - x)^2$	This *second-order reaction rate* law gives the rate at which a single chemical species combines to produce a new species, such as methyl radicals combining in a gas to form ethane molecules. See Atkins [3, p. 134].
$\frac{d^2y}{dx^2} = \frac{C}{L}\sqrt{\left(\frac{AC}{L}\right)^2 + \left(\frac{dy}{dx}\right)^2}$	The graph of the solution models the shape of the *Gateway Arch* in St. Louis, where y is its height at a distance x from one end of its base. The constants A, C, and L relate the lengths of the base, top, and centroid. The Gateway Arch has the shape of an inverted *catenary*. A ***catenary*** is a curve that has the shape of a chain suspended from two points at the same level. The equation used to design the catenary curve shape of Arch can be found at the website: *www.nps.gov/jeff*.

Differential Equation	**Situation**
$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$	This equation models the motion of a damped mass-spring system subjected to a time-dependent force $F(t)$.
$\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dx^2} + \left(E - \frac{1}{2}kx^2\right)\psi = 0$	This equation from quantum mechanics is the time-independent *Schrödinger's equation* for the one-dimensional simple harmonic oscillator. The constant $\hbar$ is defined in terms of Planck's constant h by $\hbar = h/2\pi$.
$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda(\lambda + 1)y = 0$	This equation is known as *Legendre's differential equation* and is one of several equations used for calculating the energy levels of the hydrogen atom.
$EI\frac{d^4y}{dx^4} = w(x)$	This differential equation models the vertical displacement $y(x)$ of a point located a distance x from the fixed end of a beam of uniform cross section, where $w(x)$ represents the load at x; E and I are constants.
$x'' - \varepsilon(1 - x^2)x' + x = 0$	The *van der Pol equation* models the current at time t in an electrical circuit with nonlinear resistance.
$4xy^2\left(y^{(4)}\right)^3 - 3x^4y^5\left(y''\right)^6 = \cos^9(x^{10})$	This is just one mean-looking equation concocted by the author.
$\frac{dx}{dt} = \sigma(y - x)$ $\frac{dy}{dt} = rx - y - xz$ $\frac{dz}{dt} = xy - bz$	This set of three differential equations, called the *Lorenz system,* is a overly simplified version of a complicated system of twelve equations used to model convection in the atmosphere. The Lorenz system models the chaotic rotational motion of a wheel with leaking compartments of water symmetrically positioned around its rim. See Appendix A for more information.

Even though the equations in Table 1.1 come from diverse fields, they do have some common features. The foremost feature shared by all of them is that they have at least one derivative, which is precisely what makes them differential equations in the first place! We make special note of this by formally defining what is meant by a differential equation.

Differential Equation

Definition 1.1. A ***differential equation*** is an equation that involves one or more derivatives of some unknown function or functions.

To complicate matters, there are various types of differential equations: chief among them are *ordinary differential equations*, *partial differential equations*, and *integro-differential equations*. The equations in Table 1.1 are all examples of *ordinary differential equations* because they only involve *ordinary derivatives*. Ordinary derivatives are the derivatives that we study in a first (single-variable) calculus course. *Partial differential equations* are equations involving derivatives called *partial derivatives*—how a partial derivative differs from an ordinary derivative is discussed later on, after the review of ordinary derivatives in the next section. To give us an inkling of what partial differential equations look like, here is a classic example:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

It is used to model the conduction of heat through an extremely thin metal bar, where $u(x, t)$ is the temperature at the point x in the bar at time t.

Integro-differential equations involve not only derivatives of unknown functions but also their integrals. For example, in Chapter 10 we will solve integro-differential equations that look like

$$x'(t) = f(t) + \int_0^t k(t-u)x(u)\, du.$$

This book is devoted to a study of ordinary differential equations. Even so, there will be brief forays at times into topics involving very simple partial differential equations, integral equations, and integro-differential equations.

Before we formally define what is meant by an ordinary differential equation, let's point out some features that the equations in Table 1.1 have in common. First we observe that each equation in Table 1.1 contains a single ***independent variable*** and one or more ***dependent variables***. It is a relatively simple matter to tell these two types of variables apart from the derivatives themselves, since differentiation always takes place with respect to the independent variable. Obviously then, the other variable, the one being differentiated, is the dependent variable.

Example 1.1. The first entry in Table 1.1 is the Malthusian law of population growth:

$$\frac{dp}{dt} = rp.$$

Translated into words, the equation says that the rate at which the current population p of an organism changes with respect to the time t is equal to the product of the constant r and the current population p. The time t is the independent variable and the population p is the dependent variable.

Example 1.2. The equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

in Table 1.1 models the vertical displacement of a beam. Since y is differentiated with respect to x, the independent variable is x and the dependent one is y.

The name *dependent variable* is befitting because it describes the type of variable it is: it depends in some functional way on the independent variable as prescribed by the differential equation, although this dependence is not always possible. For example,

$$y^2 + (y')^2 = -1$$

is a differential equation; even so, no real-valued function[2] can fulfill the prescript that the sum of its square and the square of its derivative is equal to a negative number.

Space on a page can be saved by replacing ***Leibniz notation***, which uses the Latin "d" for denoting derivatives, such as

$$\frac{dp}{dt}, \quad \frac{d^2x}{dt^2}, \quad \frac{d^4y}{dx^4},$$

with a shorthand notation that uses primes ($'$) or overdots ($\dot{}$) for differentiation. In ***prime notation***, the derivatives

$$\frac{dy}{dx} \quad \text{and} \quad \frac{dp}{dt}$$

are written

$$y' \quad \text{and} \quad p',$$

respectively. A shortcoming of this notation is that the independent variable is not explicitly stated.

The ***overdot notation*** is reserved for derivatives that are taken with respect to the time t. For example, $\dot{p}$ means dp/dt. Thus, the Malthusian population law

$$\frac{dp}{dt} = rp$$

in the overdot notation becomes

$$\dot{p} = rp.$$

We also have to be aware of the *orders* of the derivatives appearing in equations. The derivatives

$$\frac{dy}{dx}, \quad \dot{p}, \quad z'$$

are ***first-order derivatives***, whereas the derivatives

$$\frac{d^2x}{dt^2}, \quad y'', \quad \ddot{p}$$

[2] A function is ***real-valued*** when every evaluation of it results in a real number. Even though the function $i \sin x$, where $i^2 = -1$, satisfies the differential equation, it is a complex-valued solution, not a real-valued solution.

are ***second-order derivatives***. Of course, some differential equations have derivatives of even higher order: ***third-order derivatives*** such as

$$y''', \quad \frac{d^3x}{dt^3}$$

or ***fourth-order derivatives*** such as

$$\frac{d^4x}{dt^4}$$

or even higher. It is easy to lose track of the number of primes or overdots when the order is more than three. In such a case, it is customary to use either Leibniz notation or to use superscripts enclosed in parentheses to denote such derivatives: for example, d^4y/dx^4 or $y^{(4)}$ is preferred over y''''. The nth derivative of y with respect to x is written as $d^n y/dx^n$ or as $y^{(n)}$.

All of the previous derivatives are known as *ordinary derivatives*. When we take the ordinary derivative of a function, the term *ordinary* indicates that we are dealing with a function of a single variable. In other words, an ***ordinary derivative*** is a derivative of a function of a single independent variable with respect to that variable. The word *ordinary* qualifies the word *derivative*, distinguishing between the derivatives of single-variable calculus from the ones of multivariable calculus. Multivariable calculus deals with functions of two or more variables; their derivatives are called *partial derivatives*. They will be introduced shortly; but for now, let's review the definition and meaning of the ordinary derivative of a function.

1.1.1 Ordinary Derivatives

Let's review the meaning of an *ordinary derivative* with an example. Imagine stretching a filament-like copper wire of length 25 centimeters tautly along a straight line. Let the line serve as the x-axis and the left end of the wire designate the location of the origin. Suppose that the wire is heated unevenly in such a way that each of its points eventually reaches a constant temperature but that the temperature generally varies from point to point. Even though in reality the wire is a three-dimensional object, its very thinness suggests that variations in temperature along the y- and z-directions are negligible. Consequently, the wire may be regarded ideally as a one-dimensional mathematical object: the line segment extending from $x = 0$ cm to $x = 25$ cm. Now suppose that Table 1.2 gives temperature measurements, accurately to the ten-thousandth place, at 5-centimeter intervals along the wire.

Table 1.2: **Temperatures at Points of an Unevenly Heated Wire**

***x* (cm)**	0	5	10	15	20	25
***T* (°C)**	100.0000	99.8740	99.4980	98.8720	97.9960	96.8700

The ***average rate of change*** of the temperature with respect to x, as x changes from 10 cm to 15 cm, is given by the *difference quotient* $\Delta T/\Delta x$, where ΔT is the change in the temperature corresponding to Δx, the change in the x-coordinate. Thus,

$$\frac{\Delta T}{\Delta x} = \frac{T(15) - T(10)}{15 - 10} = \frac{98.8720 - 99.4980}{5} = -0.1252\,^\circ\mathrm{C/cm}.$$

Since the negative sign comes from ΔT, the quantity $0.1252\,^\circ$C is interpreted as the temperature decrease per centimeter (roughly) as x increases from 10 cm to 15 cm. Likewise, the difference quotient when x decreases from 10 cm to 5 cm is

$$\frac{\Delta T}{\Delta x} = \frac{T(5) - T(10)}{5 - 10} = \frac{99.8740 - 99.4980}{-5} = -0.0752\,^\circ\mathrm{C/cm}.$$

The negative sign results from the decrease in x accompanied by the increase in T. Consequently, the quantity $0.0752\,^\circ$C roughly estimates the temperature increase per centimeter as x decreases from 10 cm to 5 cm. Equivalently, we can view the temperature as decreasing roughly $0.0752\,^\circ$C per centimeter as x increases from 5 cm to 10 cm.

If no other temperature measurements are available to us, we could use one of the two previously calculated values as a rough estimate of the rate of change of the temperature near $x = 10$ cm. Better yet, we could use their average. Even so, a change of 5 centimeters in x is a big jump when it comes to estimating the rate of change in the temperature near $x = 10$ cm. Better estimates could be obtained with smaller jumps in x. For example, suppose that we are also able to measure the temperature at $x = 11$ cm. Then an estimate of the rate at which the temperature decreases starting at $x = 10$ cm is given by the difference quotient

$$\frac{\Delta T}{\Delta x} = \frac{T(11) - T(10)}{11 - 10}.$$

Suppose the measured temperature at $x = 11$ cm is $99.3928\,^\circ$C; then,

$$\frac{\Delta T}{\Delta x} = \frac{99.3928 - 99.498}{1} = -0.1052\,^\circ\mathrm{C/cm}.$$

This provides us with a new estimate of the rate at which the temperature decreases when $x = 10$ cm; namely, $0.1052\,^\circ$C per centimeter. This is an improvement over the previous estimates since Δx is now smaller by a factor of 5.

Of course, even better estimates than the previous ones could be obtained by having data for even smaller values of Δx. The ideal situation would be to know the temperature at every point of the wire. Then the *instantaneous rate of change* of the temperature at $x = 10$ cm would be given precisely by the limit that the difference quotient $\Delta T/\Delta x$ approaches as Δx approaches 0, where $\Delta x = x - 10$ and ΔT is the corresponding change in the temperature from 10 cm to x cm. In mathematical notation, we express this by writing

$$\lim_{\Delta x \to 0} \frac{\Delta T}{\Delta x} = \lim_{\Delta x \to 0} \frac{T(10 + \Delta x) - T(10)}{\Delta x}.$$

This limit is known as the *derivative* of the temperature at $x = 10$ cm and is symbolized by $T'(10)$. For example, let's suppose that the temperature at every point of the wire is given by the function

$$T(x) = 100 - 0.0002x - 0.005x^2.$$

In fact, the values in Table 1.2 were constructed using this function. Thus, the temperature change ΔT from $x = 10$ cm to $x = 10 + \Delta x$ cm is

$$\begin{aligned}\Delta T &= T(10 + \Delta x) - T(10)\\ &= 100 - 0.0002(10 + \Delta x) - 0.005(10 + \Delta x)^2 - 99.4980\\ &= -0.1002\Delta x - 0.005(\Delta x)^2.\end{aligned}$$

It follows that the corresponding difference quotient is

$$\frac{\Delta T}{\Delta x} = -0.1002 - 0.005\Delta x.$$

Hence, $\Delta T/\Delta x$ approaches $-0.1002\,°\text{C}/\text{cm}$ as Δx approaches 0. We conclude that $T'(10) = -0.1002\,°\text{C}/\text{cm}$, which means that the temperature decreases at an instantaneous rate of $0.1002\,°\text{C}$ per centimeter when $x = 10$ cm. Of course, the derivative $T'(10)$ can be obtained more easily from the derivative rules of calculus. At any value of x,

$$T'(x) = \frac{d}{dx}\big(100 - 0.0002x - 0.005x^2\big) = -0.0002 - 0.01x.$$

In particular, at $x = 10$,

$$T'(10) = -0.0002 - 0.01(10) = -0.1002\,°\text{C}/\text{cm}.$$

Now that we have reviewed the meaning of a derivative with an example, let's consider derivatives in general. It is often the case when dealing with a function, call it $f(x)$, that the instantaneous rate[3] with which it changes with respect to x needs to be determined. This rate is computed from the ***difference quotient*** for f:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

This gives the ***average rate of change*** of f from x to $x + \Delta x$. The ***instantaneous rate of change*** of f at x is the limit of the difference quotient as Δx approaches 0, which is expressed by writing

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{1.1.1}$$

provided that this limit actually exists. This limit is what is meant by the ***ordinary derivative of*** f ***at*** x. It is symbolized by $f'(x)$ or by dy/dx if y is the dependent variable. The result of (1.1.1), if the limit exists, is an expression in x that represents the ordinary derivative f'. It is as much a function of x as is f; however, its domain may differ from f's. The domain of f' consists of all x-values for which the limit (1.1.1) exists. For example, the domain of the real-valued function

$$f(x) = \sqrt{x}$$

[3] Usually the term *instantaneous* is omitted.

consists of all nonnegative numbers ($x \geq 0$). For these values of x, with the exception of $x = 0$, the difference quotient (1.1.1) converges to $1/(2\sqrt{x})$. Therefore,

$$f'(x) = \frac{1}{2\sqrt{x}}$$

and its domain consists of all positive numbers ($x > 0$). Of course, the simplest way to find derivatives is not to apply (1.1.1) directly but to use the rules of differentiation learned in elementary calculus. When x is assigned a particular value, say $x = a$, then $f'(a)$ is a number that represents the (instantaneous) rate of change of f at $x = a$.

1.1.2 Ordinary Differential Equations

Now that we have reviewed the definition of the ordinary derivative, we can state what is meant by the type of differential equation known as an ordinary differential equation.

Ordinary Differential Equation

> **Definition 1.2.** An ***ordinary differential equation*** is an equation involving one independent variable; one or more dependent variables, each of which is a function of the independent variable; and ordinary derivatives of one or more of the dependent variables.

Each of the equations in Table 1.1 fits the description in Definition 1.2; accordingly, they are all ordinary differential equations. Still, a few words need to be said about the situation described in the last entry of the table, which unlike the others requires more than one differential equation to model it, namely, the system of three equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz. \end{aligned} \tag{1.1.2}$$

When a steady, uniformly distributed shower of water falls over a wheel with leaky compartments symmetrically positioned around its rim, it can be shown that together these three equations model the rotational motion of the wheel.[4] Observe that the equations are coupled or linked together by their sharing of the three dependent variables: x, y, and z. As a result, we say that they make up a ***system of ordinary differential equations***. This system, known as the ***Lorenz system*** or the ***Lorenz equations***, is legendary in being one of the catalysts in initiating a branch of mathematics and science known as *chaos*. The equation $\dot{y} = rx - y - xz$ is still considered an ordinary differential equation, even though it contains all three dependent variables. The reason for this is that $\dot{y}$ is an ordinary derivative and all three dependent variables depend only on the single independent variable t. This is implied from the form of the three equations in

[4]For more information and a derivation of the equations, see the section entitled "Chaotic Motion in Water Wheels and the Lorenz Equations" in the Appendix.

the system. Likewise, the other two equations are also ordinary differential equations. In other words, the Lorenz system consists of three ordinary differential equations.

One characteristic of a differential equation is the order of the highest derivative appearing in the equation. For example, if a differential equation contains a derivative of second order (a second derivative) but none of higher order, then we say that the differential equation is second order or of order 2. Legendre's equation listed in Table 1.1 is a second-order equation. The Lorenz system consists of three first-order equations. Other examples are:

(a) $\dfrac{dx}{dt} = k(a - x^2)$ is an equation of order 1 (or 1*st* order);

(b) $2xyy' + (yy')^2 = y^2$ is a 1*st* order equation;

(c) $EI\dfrac{d^4y}{dx^4} = w(x)$ is an equation of order 4;

(d) $4xy^2\left(y^{(4)}\right)^3 - 3x^4y^5\left(y''\right)^6 = \cos^9(x^{10})$ is a 4*th* order equation.

Order of an Ordinary Differential Equation

Definition 1.3. The ***order*** of an ordinary differential equation is said to be n if the order of the highest derivative appearing in the equation is n.

Besides an independent variable and a dependent variable (or variables), most of the ordinary differential equations listed in Table 1.1 contain quantities known as *parameters*. A ***parameter*** does not change in value with changes in the value of the independent variable; however, its value may change when the situation or experiment is modified. For example, consider the simple differential equation

$$\frac{dp}{dt} = rp,$$

which is known as the Malthusian law when it is used to predict the populations of certain types of species. The quantity r is a constant for a given species; that is, its value does not change with time. Yet its value will most likely change if it is applied to a different species. Another example is the damped mass-spring system

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t),$$

which has three parameters: m, b, and k. The parameter m is the mass of a body to which a spring is attached. The parameter k measures the stiffness of the spring, and b measures the retardation in the motion of the body due to damping forces, such as friction. None of these parameters depends on time, yet their values would change if the body and spring were replaced by some other body and spring.

1.1.3 Solutions of Ordinary Differential Equations

Generally speaking, when dealing with a differential equation, the goal is to find out as much as possible about its solutions. However, the meaning of the word "solution" in the context of differential equations is sometimes misunderstood by students. The reason for this goes back to its meaning in algebra, trigonometry, and calculus—in these worlds, solutions are numbers. But in the world of differential equations solutions are not numbers . We illustrate by comparing a solution of an algebraic equation to that of a differential equation. The solution of

$$2x - 3 = x + 7$$

is "10", a number! Why is it a solution? The answer, of course, is that "10" satisfies this equation. Substitution of "10" for the unknown "x" results in the left- and right-hand sides of the equation being equal: the left-hand side (LHS) becomes

$$\text{LHS} = 2x - 3 = 2(10) - 3 = 17,$$

which equals the right-hand side (RHS)

$$\text{RHS} = x + 7 = 10 + 7 = 17.$$

By contrast, the solutions of differential equations are functions, not numbers. A simple example is provided by the differential equation

$$\frac{dy}{dx} = 2x.$$

A solution is $y = x^2$. In fact, every function of the form $y = x^2 + C$, where C is a constant, is a solution. The reason for this is that these functions satisfy the equation. When we substitute "$x^2 + C$" for the unknown y and differentiate, we obtain "$2x$", which is precisely the right-hand side of the equation:

$$\text{LHS} = \frac{d}{dx}y = \frac{d}{dx}(x^2 + C) = 2x \quad \text{equals} \quad \text{RHS} = 2x.$$

No other functions have derivatives equal to $2x$, aside from those of the form "$x^2 + C$". Consequently, these are the only solutions of the differential equation. Let's take a look at some more examples.

Example 1.3. Suppose someone alleges that $y = \ln x$ is a solution of

$$xy'' = -y'. \tag{1.1.3}$$

Determine whether it really is a solution.

Solution. We have not yet learned any methods for solving differential equations. But we do not need any to answer this question. All that is required is for us to substitute the first and second derivatives of $\ln x$ into the equation to see if a true statement results. With these substitutions, the left-hand side of the equation is

$$\text{LHS} = xy'' = x\frac{d^2}{dx^2}(\ln x) = x\frac{d}{dx}\left(\frac{1}{x}\right) = x\left(-\frac{1}{x^2}\right) = -\frac{1}{x},$$

while the right-hand side is

$$\text{RHS} = -\frac{d}{dx}(\ln x) = -\frac{1}{x}.$$

Since both the left- and right-hand sides turn out to be equal to $-x^{-1}$, the function $y = \ln x$ is a solution.

There is another matter to consider. Since a solution is a function, we need to be aware of the domain of its definition and where it solves the differential equation. As for this example, we know from calculus that the domain of $\ln x$ is the interval $(0, \infty)$. Since $\ln x$ also satisfies (1.1.3) on this interval, we say that the maximal interval for which $y = \ln x$ is a solution of (1.1.3) is $(0, \infty)$.

Example 1.4. The function $y = x^2$ is also alleged to be a solution of (1.1.3). Is it really?

Solution. Substituting, we find that

$$\text{LHS} = x\frac{d^2}{dx^2}(x^2) = x\frac{d}{dt}(2x) = 2x$$

whereas

$$\text{RHS} = -\frac{d}{dx}(x^2) = -2x.$$

Since the LHS $\neq$ RHS, we conclude that $y = x^2$ is not a solution of the differential equation.

Example 1.5. Is $y = 1$ a solution of (1.1.3)?

Solution. This might be interpreted as: Is "1" a solution? But that would be incorrect. The question really asks: Does the constant function $y(x) = 1$ satisfy the equation? Now this may seem like quibbling over semantics but there is a point to be made:

> *Solutions of algebraic equations are numbers. Solutions of differential equations are functions.*

Now the answer: Since both the first and second derivatives of this function are equal to 0 at all values of x, it satisfies the equation for all $-\infty < x < \infty$. Thus $y(x) \equiv 1$ is indeed a solution.[5]

Before presenting any more examples, let us summarize what the previous examples have taught us about what is meant by a *solution* of a differential equation.

[5]The symbol $\equiv$ stands for the phrase "is identically equal to." So $y(x) \equiv 1$ means $y(x) = 1$ for all $-\infty < x < \infty$.

Solution of an Ordinary Differential Equation

Definition 1.4. A ***solution*** of an ordinary differential equation with one dependent variable is a differentiable function of the independent variable that satisfies the equation on some interval. In other words, if we substitute the function for the dependent variable, we obtain a result that is valid on the interval.

Example 1.6. Is $y = \sin x$ a solution of the equation $y^2 + (y')^2 = 1$?

Solution. When we substitute $\sin x$ for y, we obtain $y^2 + (y')^2 = \sin^2 x + \cos^2 x = 1$ by the Pythagorean identity of trigonometry. Since this is true for all real numbers, $y = \sin x$ is a solution on the interval $(-\infty, \infty)$.

An algebraic equation may not have real-valued solutions, such as $x^2 = -1$. The same may be true of a differential equation. Consider, for instance,

$$y^2 + (y')^2 = -1.$$

Whatever real-valued, differentiable function is substituted for y, the left-hand side of the equation is always nonnegative. Consequently, there is no real-valued function that solves this equation.

1.1.4 Partial Derivatives

We introduce the concept of *partial derivatives* much in the same way as our review of *ordinary derivatives* by again considering temperature variations in an unevenly heated copper object. This time, however, instead of a filament-like copper wire, we imagine heating a thin, rectangular copper plate with a length of 25 cm and a width of 5 cm. The thinness of the plate allows us to ignore its thickness in the ensuing discussion; in effect, we model the real, three-dimensional plate with an idealized two-dimensional rectangle. Let's orient the plate so that two of its adjoining edges are along the x- and y-axes as shown in Figure 1.1.

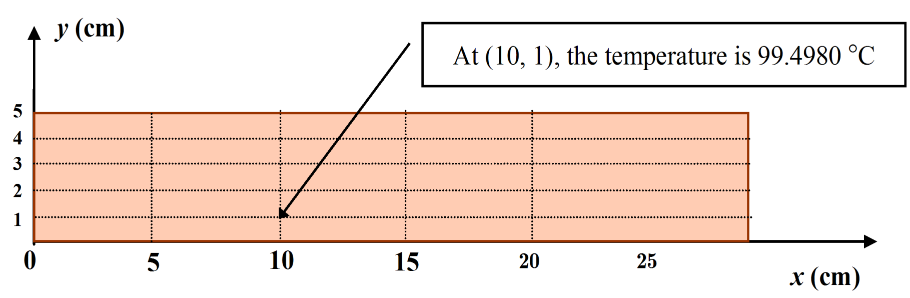

Fig. 1.1: Unevenly heated copper plate

As with the previous example of the copper wire, let's imagine that the plate is heated unevenly in such a way that the temperature at each of its points stays constant but that it generally varies from point to point. One possibility is given in Table 1.3, which gives temperatures accurately measured to the ten-thousandth place at the points (x, y). Let's use the notation $T(x, y)$ to designate the temperature at (x, y). In other words, T is the name that we give to the temperature function. Note that it depends on both x and y—in other words, it is a function of two variables rather than just one variable.

Table 1.3: **Temperatures (°C) at Points of an Unevenly Heated Plate**

		***x* (cm)**					
		0	5	10	15	20	25
***y*(cm)**	0	99.0000	98.8750	**98.5000**	97.8750	97.0000	95.8750
	1	**100.0000**	**99.8740**	**99.4980**	**98.8720**	**97.9960**	**96.8700**
	2	101.0000	100.8710	**100.4920**	99.8630	98.9840	97.8550
	3	102.0000	101.8660	**101.4820**	100.8480	99.9640	98.8300
	4	103.0000	102.8590	**102.4680**	101.8270	100.9360	99.7950
	5	104.0000	103.8500	**103.4500**	102.8000	101.9000	100.7500

Let's select a specific point on the plate, say $(10, 1)$, in order to investigate the temperature changes near it. In Table 1.3, the temperatures in the row and column containing the temperature at $(10, 1)$, namely 99.4980 °C, are boldfaced. Consider the temperature changes along this row and column. The common feature shared by the temperatures in the row is that all of them are at points with their y-coordinates equal to 1. If we restrict our attention to the $y = 1$ row, the temperature function may be viewed as a function of the single variable x. Let $r(x)$ denote its value at x. Note that $r(x)$ is the same as $T(x, 1)$. The derivative

$$r'(10) = \lim_{\Delta x \to 0} \frac{r(10 + \Delta x) - r(10)}{\Delta x} \tag{1.1.4}$$

gives the exact (instantaneous) rate of change of the temperature T at the point $(10, 1)$ when x is varied but the y-coordinate is held at the constant value 1. One of the ways to denote this derivative is by writing $T_x(10, 1)$. The limit of the difference quotient (1.1.4) written entirely in terms of T is

$$T_x(10, 1) = \lim_{\Delta x \to 0} \frac{T(10 + \Delta x, 1) - T(10, 1)}{\Delta x}. \tag{1.1.5}$$

This particular limit is called the *partial derivative of T with respect to x at the point* $(10, 1)$.

In the same way, just as we found that (1.1.5) gives the rate of change of T at the point $(10, 1)$ when the value of y is kept fixed at 1, we can find another rate of change of T at the point $(10, 1)$ by varying y but keeping $x = 10$. This means that this time the measurements in the boldfaced column (the one with the heading 10) must be used to compute the difference quotients. Let $c(y)$ denote the temperatures in this column. One possible estimate of the rate of change of T at the point $(10, 1)$, albeit a rough one, is given by the following difference quotient obtained by changing y from $y = 1$ to $y = 2$:

$$\frac{\Delta c}{\Delta y} = \frac{c(1 + \Delta y) - c(1)}{\Delta y} = \frac{c(2) - c(1)}{2 - 1} = \frac{100.4920 - 99.4980}{1} = 0.9940\,{}^\circ\mathrm{C}/\mathrm{cm}.$$

Since $c(y) = T(10, y)$, this translates to

$$\begin{aligned}\frac{\Delta T}{\Delta y} &= \frac{T(10, 1 + \Delta y) - T(10, 1)}{\Delta y} \\ &= \frac{T(10, 2) - T(10, 1)}{2 - 1} = \frac{100.4920 - 99.4980}{1} = 0.9940\,{}^\circ\mathrm{C}/\mathrm{cm}.\end{aligned} \tag{1.1.6}$$

The quantity $0.9940\,{}^\circ$C is roughly the temperature increase per centimeter as y increases from 1 cm to 2 cm when x is held fixed at 10. The exact rate of change at the point $(10, 1)$ keeping $x = 10$ is defined by the limit

$$\lim_{\Delta y \to 0} \frac{\Delta T}{\Delta y} = \lim_{\Delta y \to 0} \frac{T(10, 1 + \Delta y) - T(10, 1)}{\Delta y}. \tag{1.1.7}$$

However, there is a fly in the ointment: if we do not know the temperatures at all of the points in the plate that have their x-coordinates equal to 10, then the limit (1.1.7) can only be estimated with a difference quotient, such as the one in (1.1.6). Nevertheless, the limit does exist because there is a temperature $T(x, y)$, whether known or not, associated with every point (x, y) on the plate. The limit given by (1.1.7) is called *partial derivative of T with respect to y at the point* $(10, 1)$ and is denoted by the symbol $T_y(10, 1)$.

Suppose that the temperature at every point of the plate is given by the function [6]

$$T(x, y) = 99 + y - 0.0002xy^2 - 0.005x^2. \tag{1.1.8}$$

Let's compute the partial derivative $T_y(10, 1)$ by taking the limit of the difference quotient in (1.1.7) as follows:

$$\begin{aligned}\lim_{\Delta y \to 0} \frac{T(10, 1 + \Delta y) - T(10, 1)}{\Delta y} &= \lim_{\Delta y \to 0} \frac{\Delta y - 0.004\Delta y - 0.002(\Delta y)^2}{\Delta y} \\ &= \lim_{\Delta y \to 0} (0.996 - 0.002\Delta y) = 0.996\,{}^\circ\mathrm{C}/\mathrm{cm}.\end{aligned}$$

Therefore, $T_y(10, 1) = 0.996\,{}^\circ\mathrm{C}/\mathrm{cm}$.

The previous example serves as an introduction to the definition of a partial derivative. The rate of change of a function of two variables with respect to one of its variables, as the other one is held constant, is known as a *partial derivative* of the function.

[6] In fact, the values in Table 1.3 were actually computed with this function.

Since there are two variables, there are two first-order partial derivatives.[7] A partial derivative of a function of two variables is obtained by taking the limit of a difference quotient for that function, just as an ordinary derivative of a function of one variable is the result of taking the limit of its difference quotient. We just have to state precisely what is meant by the differences quotients for a function of two variables and how to take their limits. Using the example of the temperature function as a guide, the following definition should come as no surprise.

First-Order Partial Derivatives

Definition 1.5. Let F be a function of x and y. The ***partial derivative of*** F ***with respect to*** x, denoted by F_x, is defined by

$$F_x(x, y) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x}, \tag{1.1.9}$$

provided the limit exists. Likewise, the ***partial derivative of*** F ***with respect to*** y, denoted by F_y, is defined by

$$F_y(x, y) = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y}, \tag{1.1.10}$$

provided this limit exists.

Example 1.7. Find the partial derivative F_x of the function $F(x, y) = 5x^2 y$.

Solution. By (1.1.9),

$$\begin{aligned} F_x(x, y) &= \lim_{\Delta x \to 0} \frac{5(x + \Delta x)^2 y - 5x^2 y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5x^2 y + 10x \Delta x y + 5(\Delta x)^2 y - 5x^2 y}{\Delta x} \\ &= \lim_{\Delta x \to 0} (10xy + 5\Delta x y) = 10xy. \end{aligned}$$

We should point out that there is another way to denote partial derivatives. $F_x(x, y)$ and $F_y(x, y)$ are also expressed by writing

$$\frac{\partial}{\partial x} F(x, y) \quad \text{and} \quad \frac{\partial}{\partial y} F(x, y),$$

respectively. The ***curly d*** distinguishes partial derivatives, expressed by symbols like $\partial/\partial x$ and $\partial/\partial y$, from ordinary derivatives, which are expressed with the Latin d, such as d/dx or d/dy.

[7] There are higher-order partial derivatives too, just as there are higher-order ordinary derivatives. However, we only discuss first-order partial derivatives in this chapter. In a future chapter, we will need to talk about second-order partial derivatives but that can wait for now.

When we examine definition (1.1.9) and look at the result of the example, we see that taking the partial derivative of F with respect to x is really the same as taking the ordinary derivative of F with respect to x, if at the same time we regard y as being held at a constant value. Symbolically, we could write

$$F_x(x, y) = \frac{d}{dx} F(x, y = \text{constant}).$$

Similarly, the process of taking the partial derivative of F with respect to y can be remembered symbolically as

$$F_y(x, y) = \frac{d}{dy} F(x = \text{constant}, y).$$

With this viewpoint, it becomes a relatively simple matter to take the partial derivatives of two-variable functions, such as $F(x, y) = 5x^2 y$. The partial derivative of F with respect to x is

$$F_x(x, y) = \frac{\partial}{\partial x}\left(5x^2 y\right) = \frac{d}{dx} F(x, y = \text{constant}) = \frac{d}{dx}\left.(5x^2 y)\right|_{y=\text{constant}} = 10xy,$$

as we already determined in Example 1.7. Similarly, the partial derivative of F with respect to y is

$$\begin{aligned} F_y(x, y) &= \frac{\partial}{\partial y}\left(5x^2 y\right) = \frac{d}{dy} F(x = \text{constant}, y) \\ &= \frac{d}{dy}\left.(5x^2 y)\right|_{y=\text{constant}} = 5x^2. \end{aligned} \tag{1.1.11}$$

In practice,

$$\frac{d}{dx} F(x, y = \text{constant}) \quad \text{and} \quad \frac{d}{dy} F(x = \text{constant}, y)$$

are mental steps but are not written down. We employ this merely as a pedagogic aid for newcomers to this subject. Once you become adept at finding partial derivatives, you may think (1.1.11) but should write

$$F_y(x, y) = \frac{\partial}{\partial y}\left(5x^2 y\right) = 5x^2.$$

Example 1.8. Evaluate the first-order partial derivatives of the temperature function (1.1.8)

$$T(x, y) = 99 + y - 0.0002xy^2 - 0.005x^2 \ {}^\circ\text{C}$$

at the point $(10, 1)$.

Solution. First we find the partial derivative of F with respect to x as follows:

$$T_x(x,y) = \frac{\partial}{\partial x}T(x,y) = \frac{\partial}{\partial x}(99+y-0.0002xy^2-0.005x^2) = -0.0002y^2-0.01x.$$

Next we evaluate the result at the point $(10,1)$:

$$\begin{aligned} T_x(10,1) &= (-0.0002y^2 - 0.01x)\big|_{(10,1)} \\ &= -0.0002(1)^2 - 0.01(10) = -0.1002\,^\circ\mathrm{C/cm}. \end{aligned}$$

Similarly, the partial derivative of F with respect to y is

$$T_y(x,y) = \frac{\partial}{\partial y}T(x,y) = \frac{\partial}{\partial y}(99+y-0.0002xy^2-0.005x^2) = 1-0.0004xy.$$

Consequently,

$$T_y(10,1) = (1-0.0004xy)\big|_{(10,1)} = 1 - 0.0004(10)(1) = 0.996\,^\circ\mathrm{C/cm}.$$

If you recall, the result $T_y(10,1) = 0.996\,^\circ\mathrm{C/cm}$ was also obtained by directly applying the difference quotient. Note the ease with which we can find partial derivatives by applying the rules of differentiation that we already know from calculus.

1.1.5 Partial Differential Equations

Up to now we have explained what ordinary differential equations are and given examples. They basically are equations containing ordinary derivatives. Likewise, equations containing partial derivatives (but not ordinary derivatives) are called partial differential equations. In this book we are not concerned with partial differential equations per se; nevertheless, we will need to know a little about them. As it turns out, some methods for solving certain kinds of ordinary differential equations involve partial derivatives and some basic equations containing them. We will see this in Chapter 8. Let's begin with the definition of a so-called partial differential equation.

Partial Differential Equation

Definition 1.6. A ***partial differential equation*** is an equation containing more than one independent variable, one or more dependent variables, and partial derivatives of one or more of these dependent variables.

Our first example of a partial differential equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

It models the conduction of heat through an extremely thin metal bar, where $u(x,t)$ is the temperature at the point x in the bar at time t. This equation is known as the *one-dimensional heat equation.*[8] There are two independent variables: the spatial variable

[8] For more information, see Churchill [17].

x and the temporal variable t. The variable u depends on both of them and so it is the dependent variable. The parameter k is called the *thermal conductivity* of the bar. What makes this a partial differential equation is that it is an equation containing partial derivatives. Whereas $\partial u/\partial t$ is a first-order partial derivative, $\partial^2 u/\partial t^2$ is a second-order partial derivative, similar to second-order ordinary derivatives. Therefore, this is an example of a second-order partial differential equation. We will say a little more about higher-order partial derivatives in a later chapter.

The only partial differential equations that we need to consider in this book are first-order equations of the form

$$\frac{\partial u}{\partial x} = f(x, y) \quad \text{or} \quad \frac{\partial u}{\partial y} = g(x, y), \tag{1.1.12}$$

where the dependent variable u is a function of both x and y. An example of an equation of the first form is

$$\frac{\partial u}{\partial x} = 2xy - \sin x.$$

An example of one of the second form is

$$\frac{\partial u}{\partial y} = 5y^4 + \frac{2xy}{x^2 + y^2} + 10.$$

In Section 1.2.3, we will discuss how to find solutions of partial differential equations of the two forms shown in (1.1.12). For the sake of brevity, it is common to use the abbreviations "ODE" for "ordinary differential equation" and "PDE" for "partial differential equation."

1.2 Origins of Basic Ordinary Differential Equations

As Section 1.1 points out, ordinary differential equations arise when we attempt to use mathematics to model certain real-life situations. Equations that are judged good models imitate reality closely, provide insight and understanding, and predict well. Finding just the right equation, or equations as the case may be, could be quite complicated—not only because of the mathematics but because other disciplines are involved as well. In spite of this, we will ease our way into a study of differential equations by starting with some of the more elementary ones that arise from modeling relatively simple situations. Throughout the rest of this chapter, we present examples of elementary differential equations that result when

(a) the rate of change of a quantity is known or can easily be determined;

(b) the rate of change of a quantity is known or conjectured to be proportional to another quantity;

(c) Newton's second law of motion is used to model the motion of a body.

1.2.1 Rates of Change

Modeling real-life situations frequently involves the rates of change of quantities. From studying calculus, we have learned that the (***instantaneous***) ***rate of change of a quantity***, dependent solely on a single variable, is given by its ordinary derivative with respect to the variable. As a result, when the rate of change of a quantity is given or can be determined, that information can be expressed mathematically as an ordinary differential equation. Let's take a look at some examples of typical situations involving rates of change.

Bathtub

Water flows out of a spout into a bathtub at the rate of 3 gallons per minute. Translated in the succinct language of mathematics, this verbal statement becomes the differential equation

$$\frac{dN}{dt} = 3,$$

where $N(t)$ denotes the number of gallons of water that has flowed into the bathtub after t minutes.

Temperature along a heated wire

In our review of the meaning of an ordinary derivative, we determined that the temperature T along the copper wire changes at a rate of $-0.0002 - 0.01x$ degrees Celsius per centimeter, where x is the distance in centimeters from the left end of the wire. This verbal statement is equivalent to the mathematical statement

$$\frac{dT}{dx} = -0.0002 - 0.01x.$$

Marginal cost

An economic decision may be based in part on the increment in cost that will be incurred if one more unit of some product is manufactured. Some economists call this cost increment the *marginal cost*. It can usually be approximated by a derivative: the instantaneous rate of change of the total cost function with respect to the number of manufactured units, say x, of the product. For this reason, many economists prefer defining the ***marginal cost*** (abbreviated *MC*) as this derivative. In this book, we use the derivative definition of marginal cost. As an example, suppose it is stated that the marginal cost to manufacture x widgets[9] is given by the function $MC = 10x + 5000$. We can express this statement more concisely with the differential equation

$$\frac{dC}{dx} = 10x + 5000,$$

where $C(x)$ denotes the total cost to produce x widgets.

[9] A ***widget*** is a substitute for the name of some device or gadget, usually used when its real name is not known or temporally forgotten, in other words, a thingamajig. Here we use it to mean a fictitious, manufactured product. In this way, we can easily fabricate marginal cost functions to illustrate the mathematics and economics, without also having to consider whether or not they represent reality.

Speed of a falling object

An object is dropped from the top story of the Leaning Tower of Pisa, the famous freestanding, eight-story bell tower of the cathedral of Pisa, Italy. Let s denote the distance, in feet, the object has fallen t seconds after being dropped. The ***speed*** of the falling object is the rate with which its distance increases with time, or the time derivative $\dot{s}$. According to the laws of elementary mechanics, the speed after t seconds is approximately $32t$ feet per second, provided the counteracting resistance of the air pushing upward on the object is negligible. Therefore, the speed of the falling object, up to the time it hits the ground, is modeled by the differential equation $\dot{s} = 32t$, or

$$\frac{ds}{dt} = 32t.$$

Arc Length

A pair of equations, $x = f(t)$ and $y = g(t)$, will generate a curve in the xy-plane when the independent variable t increases from a starting value α to an end value β. Under the assumption that the functions f and g have continuous derivatives, let's investigate how one goes about calculating the rate at which the length of the curve increases with t.

Much of elementary calculus deals with graphs of *smooth* functions. A function $F(x)$ defined for $\alpha \le x \le \beta$, where α and β are real numbers, is said to be ***smooth*** if $F'(x)$ exists and is continuous at every point of $[\alpha, \beta]$. The graph of $y = F(x)$ when F is smooth is called a ***smooth curve***. Each value of x corresponds to a point $(x, y) = (x, F(x))$ on the curve. As x increases from α to β, we can envision a point particle starting out at the *initial point* $(\alpha, F(\alpha))$ of the curve and moving along it until it ends up at the *terminal point* $(\beta, F(\beta))$.

Similarly, a pair of equations

$$x = f(t), \ \ y = g(t) \tag{1.2.1}$$

for $\alpha \le t \le \beta$ traces out a curve in the xy-plane as the independent variable t varies from $t = \alpha$ to $t = \beta$. Unlike before, the variable x is no longer an independent variable; rather it, as does y, depends on the independent variable t. Each value of t corresponds to a point $(x, y) = (f(t), g(t))$ on the curve. The equations $x = f(t)$ and $y = g(t)$ defining x and y are called ***parametric equations***. The variable t is the ***parameter***. A curve defined by (1.2.1) is said to be ***smooth*** if the derivatives of f and g exist, are continuous, and never simultaneously zero for all values of t in the interval $[\alpha, \beta]$. Smoothness guarantees that

1. there is a unique tangent line at every point of the curve, and
2. $\theta(t)$, the angle of inclination of the tangent line at the point $(f(t), g(t))$, is defined for all values of $t \in [\alpha, \beta]$ and is a continuous function.

Consequently, a smooth curve has no corners or cusps. Moreover, no portion of the curve is retraced as t increases. In other words, a point particle starting out at the *initial point* $(f(\alpha), g(\alpha))$ will never stop and move in the reverse direction along the curve before it reaches the *terminal point* $(f(\beta), g(\beta))$. With that said, we are now ready to consider the length of a smooth curve, which is called its ***arc length***, and to determine the rate at which its arc length changes with respect to t.

Let s denote the arc length of a curve. For $t_1 \geq \alpha$, let Δs denote the length of the portion of the curve corresponding to values of t from $t = t_1$ to $t = t_1 + \Delta t$. For values of Δt close to zero, the length Δs of this portion is approximately the length of the hypotenuse of the right triangle as illustrated in Figure 1.2

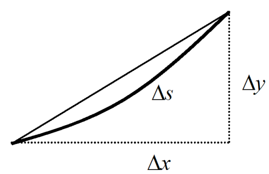

Fig. 1.2: Approximation of Δs

By the Pythagorean theorem,

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Since the difference quotient

$$\frac{\Delta x}{\Delta t} = \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}$$

is approximately equal to the derivative $f'(t_1)$ for values of Δt near zero, we can approximate Δx with $f'(t_1)\Delta t$. Likewise, $\Delta y \approx g'(t_1)\Delta t$. Thus,

$$\Delta s \approx \sqrt{(f'(t_1)\Delta t)^2 + (g'(t_1)\Delta t)^2} = \Delta t \sqrt{(f'(t_1))^2 + (g'(t_1))^2},$$

or

$$\frac{\Delta s}{\Delta t} \approx \sqrt{(f'(t_1))^2 + (g(t_1))^2}.$$

As $\Delta t \to 0$, this approximation becomes better and better. We conclude that the rate of change of the arc length s of the curve at $t = t_1$ can be found by evaluating

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad (1.2.2)$$

at $t = t_1$.

In applications where t represents the time and $(x, y) = (f(t), g(t))$ the position of a moving body at time t, the derivative $\dot{s}$ given by (1.2.2) is the rate of change of the distance of the body from its initial position, i.e., the ***speed*** of the particle.

1.2.2 The Simplest Type of Ordinary Differential Equation

Before we show examples of differential equations arising from proportions or from Newton's second law of motion, let's first note the form of the previous equations and then look into a method for finding their solutions. Each of them was the result of knowing the rate at which a quantity changes. Their form is simply a derivative equal to a constant or to an expression involving the independent variable but not the dependent one. In mathematical notation, this is expressed as

$$\frac{dy}{dx} = f(x). \tag{1.2.3}$$

The notation $f(x)$ conveys that f is a function of x but not of y while dy/dx tells us that x is the independent variable and y is the dependent variable. It is important to keep in mind that the right-hand side of (1.2.3) is a function of the independent variable alone. Whatever is said in this section about finding solutions of (1.2.3) does not carry over to equations of the form

$$\frac{dy}{dx} = f(y),$$

where f is a function of the dependent variable alone, nor to equations of the form

$$\frac{dy}{dx} = f(x, y),$$

where f is a function of both variables.

Equations of the form shown in (1.2.3) are the simplest type of ordinary differential equation in the sense that all that is required to find their solutions is direct integration with respect to x. A ***solution*** of (1.2.3) is any function whose derivative is the function f. Calculus tells us that there are infinitely many such solutions–any two of which differ by a mere constant—and this infinite set is known as the ***indefinite integral of the function*** f, which is symbolized by

$$y = \int f(x)\,dx. \tag{1.2.4}$$

That is to say, all of the solutions of (1.2.3) are the same as the set of all antiderivatives of the function f. For example, consider the differential equation

$$\frac{dy}{dx} = 2x.$$

In the general notation of (1.2.3), $f(x) = 2x$. All solutions make up the set of all antiderivatives of $2x$; namely, $x^2 + C$ as

$$\frac{d}{dx}(x^2 + C) = 2x.$$

In other words, all solutions are computed from (1.2.4). Thus,

$$y = \int f(x)\,dx = \int 2x\,dx = x^2 + C.$$

Sometimes we may have to reach a little further back into the recesses of our memories of calculus to come up with the antiderivatives of a function, or we may have to refer to integral tables. For example, the solutions of the differential equation

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

are the functions

$$y = \int \frac{dx}{1+x^2} = \tan^{-1} x + C,$$

since

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

In addition to tables for the purpose of looking up formulas of indefinite integrals, we have technological means available to us as well, such as computer algebra systems[10] or certain advanced hand-held calculators that are really calculator-and-computer hybrids. Even so, those integrals that frequently arise in elementary applications and are regarded as **standard integral forms**[11] should be known and their formulas memorized. These integrals occur often enough to be memorized simply for the sake of efficiency and time! Moreover, they are so well-known that it could prove to be somewhat embarrassing if they are not committed to memory. For instance, would you not question your professor's competency to teach this course if he or she had to resort to a table of integrals, a computer, or a calculator to find the formula for $\int x^2\,dx$? This is not to mention the time wasted and effort involved in locating the integral in the table or in turning a computer on and entering the appropriate commands.

In Table 1.4 we list the integral formulas for the standard integral forms that will turn up regularly in the text and problem sets and which a student of calculus and differential equations should have memorized.[12] For a function $f(u)$ listed in the table, $\int f(u)\,du$ is its indefinite integral. As you recall from calculus, the indefinite integral is evaluated by finding an antiderivative of $f(u)$ and adding a constant of integration, say C, to it. The result is an expression that represents all of the antiderivatives of $f(u)$. Equivalently, this represents all solutions of the differential equation

$$\frac{dy}{du} = f(u).$$

[10] A ***computer algebra system*** or ***CAS*** is an interactive computer program that can carry out mathematical computations with symbolic expressions, such as yielding the result x^2 when it is given a command to compute the integral of $2x$. Some well-known computer algebra systems are *Maple*, *Mathematica*, *Derive*, and *MATLAB*. Even some calculator models, such as Texas Instruments' *TI-89*, have software that perform symbolic manipulations.

[11] Also called **elementary forms**, as in the *Handbook of Chemistry and Physics*.

[12] By memorizing the formulas in Table 1.4, you will be able to work 90% or better all of the problems in this textbook without having to resort to integral tables, calculators, or computer algebra systems.

Table 1.4: **Standard Integral Formulas**

$\boldsymbol{f(u)}$	$\boldsymbol{\int f(u)\,du}$
$k \quad (k$, a constant$)$	$ku + C$
$u^n \qquad (n \neq -1)$	$\dfrac{u^{n+1}}{n+1} + C$
$u^{-1} = \dfrac{1}{u}$	$\ln\|u\| + C = \begin{cases} \ln u + C, & \text{if } u > 0 \\ \ln(-u) + C, & \text{if } u < 0 \end{cases}$
e^u	$e^u + C$
$\cos u$	$\sin u + C$
$\sin u$	$-\cos u + C$
$\sec^2 u$	$\tan u + C$
$\csc^2 u$	$-\cot u + C$
$\sec u$	$\ln\|\sec u + \tan u\| + C$
$\csc u$	$-\ln\|\csc u + \cot u\| + C$
$\sec u \tan u$	$\sec u + C$
$\csc u \cot u$	$-\csc u + C$

$f(u)$	$\int f(u)\,du$
$\dfrac{1}{1+u^2}$	$\tan^{-1} u + C$
$\dfrac{1}{\sqrt{1-u^2}}$	$\sin^{-1} u + C$
$\dfrac{1}{u\sqrt{u^2-1}}$	$\sec^{-1} \lvert u\rvert + C$

1.2.3 The Simplest Types of Partial Differential Equations

In the previous section, we saw that direct integration with respect to x solves ordinary differential equations of the type

$$\frac{dy}{dx} = f(x).$$

Likewise, integration with respect to a single variable is all that is required to find solutions of equations of the following two types:

$$\frac{\partial u}{\partial x} = f(x, y) \tag{1.2.5}$$

and

$$\frac{\partial u}{\partial y} = f(x, y). \tag{1.2.6}$$

These are the simplest types of partial differential equations.

A solution of (1.2.5) is a function whose partial derivative with respect to x is equal to $f(x, y)$. Solving (1.2.5) means finding all of the functions that satisfy (1.2.5). Recall that partial differentiation with respect to x is simply ordinary differentiation with respect to x with y held fixed. Solutions are found by reversing the operation of differentiation, that is, by integration. Since differentiation with respect to x is carried out by holding y fixed, the inverse operation of integration is carried out by integrating with respect to x holding y fixed. We express this symbolically in this way:

$$u = \int f(x, y)\,dx.$$

As an example, let's find solutions of the partial differential equation

$$\frac{\partial u}{\partial x} = 2xy - \sin x. \tag{1.2.7}$$

Integrating with respect to x, we obtain the solutions

$$u = \int (2xy - \sin x)\,dx = x^2 y + \cos x + C.$$

Or so we might think! But let's reconsider. Are these really all of the solutions of (1.2.7)? When we think about it, the so-called constant of integration "C" is really more than just a constant here—any function that depends only on y should be included too. The reason of course is that the partial derivative of a function of y (but not x) with respect to x is equal to 0; if g denotes such a function, we write

$$\frac{\partial}{\partial x} g(y) = 0.$$

Therefore, the complete set of solutions of (1.2.7) include all functions of the form

$$u(x, y) = x^2 y + \cos x + g(y).$$

Finally, we check this statement by substituting $x^2 y + \cos x + g(y)$ for the dependent variable u to verify that equation (1.2.7) is satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^2 y + \cos x + g(y)\right] = 2xy - \sin x.$$

For an example of a partial differential equation of the form (1.2.6), let's replace $\partial u/\partial x$ with $\partial u/\partial y$ in (1.2.7) to obtain

$$\frac{\partial u}{\partial y} = 2xy - \sin x.$$

To reverse the partial differentiation this time, we have to integrate with respect to y. We then obtain the solutions

$$u(x, y) = \int (2xy - \sin x)\, dy = xy^2 - y \sin x + h(x),$$

where h denotes a function of x alone.

As a final example, let's return to the copper plate shown in Figure 1.1. Instead of being given a function that models the temperature variation of the plate and then being asked to find the rates at which the temperature changes along the x and y directions, suppose that we are given one of these rates and asked to find the temperature function.

Example 1.9. The rate at which the temperature of the copper plate in Figure 1.1 changes with respect to y is

$$\frac{\partial T}{\partial y} = 1 - 0.0004xy \ {}^\circ\mathrm{C/cm}$$

Find out what can be said about the temperature function T itself.

Solution. Since the differentiation of the temperature function T is with respect to y, we must integrate the right-hand side of the equation with respect to y to find T. Thus,

$$T(x, y) = \int (1 - 0.0004xy)\, dy = y - 0.0002xy^2 + h(x). \tag{1.2.8}$$

Note however that this is all that we can say about the temperature function since we are given no other information. That is, we cannot completely determine it since there is no other information that allows us to find the function h. To illustrate how we would go about finding h had we more information, suppose that we also know that temperatures along the $y = 1$ line are given by the function

$$\varphi(x) = 100 - 0.0002x - 0.005x^2.$$

If (1.2.8) is to model the temperature at all points of the plate, then it follows that $T(x, 1) = \varphi(x)$; accordingly,

$$1 - 0.0002x^2 + h(x) = 100 - 0.0002x - 0.005x^2.$$

Solving for h, we have

$$h(x) = 99 - 0.005x^2.$$

Finally, substituting this into (1.2.8) gives the temperature function[13]

$$T(x, y) = 99 + y - 0.0002xy^2 - 0.005x^2.$$

1.2.4 Basic Integration Techniques

Complete familiarity with the standard integrals and a facility with integration techniques will be deciding factors in your success, or lack thereof, in solving differential equations. Table 1.4 is a list of some standard integrals. This is not a complete list, but these particular integrals are used throughout this book. Next to the functions in the left-hand column are their antiderivatives in the right-hand column. These standard integrals must be thoroughly memorized. Unfortunately, however, an integral involved in solving a differential equation probably will not look exactly like any of those listed in Table 1.4: in that case, *substitution*, *integration by parts*, and *partial fractions* become indispensable. They are techniques for transforming integrals to one of the standard integrals. We review briefly these three integration techniques by presenting some examples. If you find yourself rusty in the use of these techniques, it would be a very good idea to refresh your memory by opening your favorite calculus book and getting out past calculus class notes and worked-out problem sets.

Example 1.10. Find all solutions of the differential equation

$$\frac{dy}{dx} = \frac{5}{x^2 + 9}.$$

Solution. Referring to (1.2.3), $f(x) = 5/(x^2 + 9)$. The solutions are given by (1.2.4). So

$$y = \int f(x)\,dx = \int \frac{5}{x^2 + 9}\,dx = 5\int \frac{1}{9 + x^2}\,dx.$$

[13] Note this example is Example 1.8 worked backwards.

This integrand most nearly resembles the integrand $1/(1+u^2)$ in Table 1.4. Hence, we need to rewrite the denominator of the integrand so that it leads off with the digit 1 instead of with 9. Factoring out the 9 achieves this and we have

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{9\left[1+\left(\frac{x}{3}\right)^2\right]}\,dx = \frac{1}{9}\int \frac{1}{1+u^2}\,3\,du,$$

where the substitution $u = x/3$ was made. Since $du/dx = 1/3$, the differential dx must be replaced with $3\,du$. Therefore, the solutions are

$$y = \frac{5}{3}\int \frac{du}{1+u^2} = \frac{5}{3}\tan^{-1} u + C = \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Example 1.11. Find the solutions of $\dfrac{dy}{dx} = \dfrac{5x}{x^2+9}$.

Solution. The difference between this integrand and the previous one is the x in the numerator. What we should note is that aside from a numerical factor the numerator is the derivative of the denominator. A standard integral results by merely substituting another variable for x^2+9, say z. Then, $dz = 2x\,dx$ and

$$y = \int \frac{5x}{x^2+9}\,dx = \frac{5}{2}\int \frac{2x\,dx}{x^2+9} = \frac{5}{2}\int \frac{dz}{z} = \frac{5}{2}\ln|z| + C.$$

Thus, the solution of the differential equation is

$$y = \frac{5}{2}\ln\left(x^2+9\right) + C.$$

Example 1.12. Find the solutions of $\dfrac{dy}{dx} = \dfrac{x\,dx}{x^2+2x+1}$.

Solution. Since the independent variable is x and the right-hand side of the equation involves it but not y, we start off by indicating the solution of the equation is

$$y = \int \frac{x}{x^2+2x+1}\,dx.$$

Now we have to figure out how to carry out the integration. As a rule of thumb, we always try substitution first. Clearly, the only possibility is $u = x^2+2x+5$. This will not work however since the differential is

$$du = (2x+2)\,dx = 2(x+1)\,dx,$$

but the numerator is not a constant multiple of $x+1$. At this point, we may be at a loss of what to do next—until we notice that the denominator can be expressed as the perfect square $(x+1)^2$. Perhaps this slight change will allow us to integrate. Now the form of the right-hand side of

$$y = \int \frac{x}{(x+1)^2}\, dx$$

suggests that we try the substitution $u = x+1$. Then $du = dx$ and $x = u-1$. Hence,

$$y = \int \frac{u-1}{u^2}\, du = \int \left(u^{-1} - u^{-2}\right) du = \ln|u| + \frac{1}{u} + C.$$

Therefore,

$$y = \ln|x+1| + \frac{1}{x+1} + C.$$

Example 1.13. Find the solutions of $\dfrac{dy}{dx} = \dfrac{x\,dx}{x^2+2x+5}$.

Solution. This is similar to Example 1.12; again it is clear that a direct substitution will not work. Unlike the previous example though, the denominator is not a perfect square. However, we can change $x^2 + 2x$ into a perfect square by completing its square. This is accomplished by adding 1, which is obtained by squaring half the coefficient of x, namely 1. Thus,

$$x^2 + 2x + 5 = (x^2 + 2x + 1) - 1 + 5 = (x+1)^2 + 4;$$

and so

$$y = \int \frac{x}{(x+1)^2 + 4}\, dx.$$

Now let $u = x+1$. Then

$$\begin{aligned} y = \int \frac{u-1}{u^2+4}\, du &= \int \frac{u}{u^2+4}\, du - \int \frac{1}{u^2+4}\, du \\ &= \frac{1}{2}\int \frac{2u}{u^2+4}\, du - \frac{1}{4}\int \frac{1}{1+\left(\frac{u}{2}\right)^2}\, du \\ &= \frac{1}{2}\ln\left(u^2+4\right) - \frac{1}{4}\tan^{-1}\left(\frac{u}{2}\right) + K \\ &= \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{4}\tan^{-1}\left(\frac{x+1}{2}\right) + K. \end{aligned}$$

Example 1.14. Find the solutions of $\dfrac{dy}{dx} = x\cos 2x$.

Solution. Once again, as the right-hand side of the equation depends only on x, solutions are given by

$$y = \int x\cos 2x\, dx.$$

Integration of the product of a power function and a sinusoidal function (sine or cosine function) calls for the method of integration by parts. Recall the ***integration by parts formula***:

$$\int u\,dv = uv - \int v\,du. \tag{1.2.9}$$

To apply the formula to $\int x\cos 2x\,dx$, we let

$$u = x, \qquad dv = \cos 2x\,dx$$

$$du = dx\ \left(\text{since } \frac{du}{dx} = 1\right), \quad v = \frac{1}{2}\sin 2x\ \left(\text{as} \int \cos 2x\,dx = \frac{1}{2}\sin 2x + K\right).$$

Then

$$y = uv - \int v\,du = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{x}{2}\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + C,$$

and so all solutions are given by

$$y = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.$$

Learning when and how to use the integration by parts formula, to wit (1.2.7), is an important skill to master. If the original integral, denoted by $\int u\,dv$, is difficult or impossible to integrate, the purpose of (1.2.7) is to present an alternative: the integral $\int v\,du$. The art of integrating by parts is in the selection of u and dv so that $\int v\,du$ is easier to integrate than is $\int u\,dv$. Achieving this is a matter of common sense, trial and error, and experience. However, if you have some difficulty in selecting u and dv the acronym *LIATE*[14] is a useful device. ***LIATE*** is a mnemonic for remembering five types of functions in the following order:

<u>L</u>ogarithmic, *<u>I</u>nverse trig*, *<u>A</u>lgebraic*, *<u>T</u>rig*, *<u>E</u>xponential*.

It is precisely this order that makes *LIATE* work. When the integrand is the product of any two of these types of functions and the substitution technique doesn't work, the integration by parts formula should be tried. Let the type appearing first in the order given by *LIATE* be u and the other one, along with the differential next to the integrand, be dv. Then apply the integration by parts formula. Hopefully, $\int v\,du$ will then be easier to integrate than $\int u\,dv$. To illustrate this, consider the integral $\int x\cos 2x\,dx$ from the previous example. Its integrand is the product of the two functions x and $\cos 2x$, where x is an algebraic function and $\cos 2x$ is a trigonometric function. Thus, we let $u = x$ because the type *Algebraic* comes before the type *Trig* in *LIATE*. So, $dv = \cos 2x\,dx$. Let's illustrate *LIATE* again by solving the next equation.

Example 1.15. Find the solutions of $\dfrac{dy}{dx} = \sin^{-1}(2x)$.

[14] See Kasube [32, pp. 210–211].

Solution. Solutions are all of the antiderivatives of $\sin^{-1}(2x)$ indicated by the indefinite integral

$$y(x) = \int \sin^{-1}(2x)\,dx.$$

Obviously, the substitution $u = 2x$ won't lead anywhere unless the formula for

$$\int \sin^{-1} u\,du$$

is already available.[15] At first, it looks as though the integration by parts technique is also useless here by virtue of the absence of a product of two functions. Nevertheless, there is one—if we use the artifice of writing the integrand as the product: $1 \cdot \sin^{-1}(2x)$. Then $u = \sin^{-1}(2x)$, since the constant function "1" belongs to the *Algebraic* type and is preceded by the *Inverse Trig* type in *LIATE*. This forces *dv* to be the rest of the integrand or "1" and the differential *dx*; that is, $dv = 1\,dx = dx$. Hence,

$$du = u'(x)\,dx = \left(\frac{d}{dx}\sin^{-1}(2x)\right)dx = \frac{2}{\sqrt{1-(2x)^2}}\,dx; \quad v = x.$$

By (1.2.9), the solutions are

$$\begin{aligned} y(x) &= \int \underbrace{\sin^{-1}(2x)}_{u}\ \underbrace{dx}_{dv=1\,dx} \\ &= \underbrace{\sin^{-1}(2x)}_{u}\ \underbrace{x}_{v} - \int \underbrace{x}_{v}\ \underbrace{\frac{2}{\sqrt{1-(2x)^2}}\,dx}_{du} \\ &= x\sin^{-1}(2x) - \left(\frac{2}{-8}\right)\int \frac{-8x\,dx}{\sqrt{1-4x^2}} = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C. \end{aligned}$$

Example 1.16. Find the solutions of $\dfrac{dy}{dx} = \dfrac{6}{x^2-9}$.

Solution. Although the right-hand side of this equation does not appear in Table 1.4, the factorability of $x^2 - 9$ is the tip-off that we can integrate using the method of partial fractions. Generally speaking, a quadratic polynomial, such as $x^2 - 9$, is said to be ***reducible over the reals*** or ***factorable*** when it can be expressed as a product of two linear polynomials with real coefficients. Otherwise, it is said to be ***irreducible over the reals***. For the sake of brevity, we will omit the phrase "over the reals." Thus, $x^2 - 9$ is reducible since it reduces to the product of the linear polynomials $x - 3$ and $x + 3$. That is, it is equal to the product $(x-3)(x+3)$. On the other hand, $x^2 + 9$ is irreducible,[16]

[15] Of course, this integral could be found in some handbook or with a computer algebra system or a sophisticated calculator. But the point of this section is to review integration and to hone the skills already acquired in a calculus course to solve relatively simple integrals without having to resort to integral tables or to technology.

[16] That is, it is irreducible over the reals (meaning the set of real numbers). However, it is reducible over the set of complex numbers since $x^2 + 9 = (x-3i)(x+3i)$, where i is defined by $i^2 = -1$.

since it is just not possible to write it as a product of two linear polynomials with real coefficients. An important result of algebra states that it is always possible to express a nonconstant polynomial with real coefficients as a product of linear and irreducible quadratic polynomials with real coefficients. So, as a rule of thumb, when faced with integrating a *rational function*,[17] first factor its denominator into a product of linear and irreducible quadratic polynomials—unless the numerator is a constant multiple of the derivative of the denominator; in which case the method of substitution will work, as demonstrated in Example 1.11.

Now let's apply the method of partial fractions to evaluate the integral

$$\int \frac{6}{x^2 - 9}\, dx.$$

Its integrand is a rational function with a reducible denominator. After factoring the denominator, we expand the integrand writing it as a sum of two simpler rational functions with linear polynomials as their denominators:

$$\frac{6}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}. \tag{1.2.10}$$

The fractions

$$\frac{A}{x-3} \quad \text{and} \quad \frac{B}{x+3}$$

are called ***partial fractions*** because their denominators contain part of the original denominator $x^2 - 9$ but not all of it. The expansion of the integrand in (1.2.10) is known as its ***partial fraction expansion***. The coefficients A and B are called ***undetermined coefficients*** until values are found making (1.2.10) an identity. To determine the values of A and B, clear out the denominators in (1.2.10) by multiplying both sides by the denominator of the integrand with the result

$$A(x+3) + B(x-3) = 6.$$

Now we can quickly find the value of A by setting $x = 3$ since this eliminates the term involving B:

$$x = 3 \quad \Rightarrow \quad 6A + 0B = 6 \quad \Rightarrow \quad A = 1.$$

Likewise, setting $x = -3$ eliminates the term involving A and yields $B = -1$. Thus,

$$\frac{6}{x^2 - 9} = \frac{1}{x-3} - \frac{1}{x+3}.$$

Now integration is a piece of cake:

$$\begin{aligned} y(x) &= \int \frac{6}{x^2 - 9}\, dx = \int \frac{dx}{x-3} - \int \frac{dx}{x+3} \\ &= \ln|x-3| - \ln|x+3| + C = \ln\left|\frac{x-3}{x+3}\right| + C. \end{aligned}$$

[17] Recall that a ***rational function*** is the quotient of two polynomials. Constants, such as 6 and π, are also considered polynomials.

Let's review next the method of partial fractions when the denominator of the rational function is the product of a linear factor and an irreducible quadratic factor.

Example 1.17. Find the solutions of $\dfrac{dy}{dx} = \dfrac{18}{x^3 + 9x}$.

Solution. The method of substitution is clearly of no help in integrating the right-hand side directly due to the absence of the factor

$$\frac{d}{dx}(x^3 + 9x) = 3x^2 + 9$$

in the numerator. Let's see how the method of partial fractions fares. First, we must factor the denominator completely. Factoring out an x, we have

$$x^3 + 9x = x(x^2 + 9).$$

No more factoring is possible, since the quadratic factor is irreducible. Since the denominator consists of two factors, the partial fraction expansion of the integrand consists of two fractions. The form of partial fractions can be summed up by the dictum:

> *Put constants over linear factors and linear factors over irreducible quadratic factors.*

As a result, the form of the expansion is

$$\frac{18}{x^3 + 9x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}.$$

We clear out the denominators of the expansion by multiplying both of its sides by $x(x^2 + 9)$, obtaining

$$A(x^2 + 9) + (Bx + C)x = 18.$$

Grouping like terms together gives

$$(A + B)x^2 + Cx + 9A = 18.$$

This equation is satisfied for all values of x if A, B, and C have values satisfying the equations:

$$A + B = 0; \quad C = 0; \quad 9A = 18.$$

Consequently, $A = 2$; $B = -2$; $C = 0$ and so

$$y = \int \frac{18}{x^3 + 9x}\,dx = \int \frac{2}{x}\,dx - \int \frac{2x}{x^2 + 9}\,dx = 2\ln|x| - \ln|x^2 + 9| + C.$$

Therefore, the solutions are $y = \ln\left(\dfrac{x^2}{x^2 + 9}\right) + C$.

We conclude this section with an example of solving a partial differential equation.

Example 1.18. Find the solutions of $\dfrac{\partial z}{\partial x} = y \sec(3x)$.

Solution. The equation itself indicates that the variable z is dependent on both x and y. The nature of the partial derivative indicates that we have to integrate with respect to x: The equation itself indicates that the variable z is dependent on both x and y. The nature of the partial derivative indicates that we have to integrate with respect to x:

$$z = \int y \sec(3x)\, dx = y \int \sec(3x) \cdot \frac{1}{3} \cdot 3\, dx = \frac{y}{3} \int \sec v\, dv,$$

where $v = 3x$. From Table 1.4, we have

$$z = \frac{y}{3} \ln|\sec v + \tan v| + g(y).$$

Therefore, the solutions of the partial differential equation are

$$z = \frac{y}{3} \ln|\sec(3x) + \tan(3x)| + g(y).$$

1.2.5 Proportions Involving Rates of Change

The statement "u ***is (directly) proportional to*** v" means that

$$u = kv$$

for some constant k. The parentheses enclosing the word "directly" indicate that its use is optional; it is frequently omitted. The constant k is called the ***constant of proportionality***. That u is proportional to v is also indicated by writing

$$u \propto v.$$

Example 1.19. The assertion that "a city's average daily garbage collection G is proportional to its population p" translates to the mathematical statement:

$$G \propto p,$$

which means

$$G = kp$$

for some constant k. If there is any truth to this statement, then the value of k would have to be determined experimentally by comparing the amount of garbage and the population of a city on a given day.

Example 1.20. Consider the wry comment of some good-natured hostess, as she watched a guest's toast land on her newly laid carpet, that the likelihood of toast landing jelly-side down is directly proportional to the cost of the carpet. With L denoting the likelihood, or probability, of this happening and C the cost, this comment translates to the mathematical statement:

$$L = \gamma C,$$

where γ is the constant of proportionality.[18]

Example 1.21. Finally, in a more serious vein, we mention a postulate by the famous physicist Max Planck that aided in the development of a field of physics called quantum mechanics. In 1900, in an attempt to reconcile the theory of black body radiation with experimental results, Planck postulated that energy is not radiated continuously but rather in discrete amounts called ***quanta*** and that a quantum of energy E is proportional to the frequency ν of the radiation.[19] That is,

$$E = h\nu$$

where h denotes the proportionality constant. By adjusting the value of h, he was able to reconcile his theory with experimental results. In fact, h is now called ***Planck's constant***. Its current accepted value is 6.63×10^{-34} joule·second.

If two variables u and v are not directly proportional but are related by the equation

$$u = \frac{k}{v}$$

for some constant k, then we say "u ***is inversely proportional to*** v" or "u ***varies inversely with*** v."

Example 1.22. In the kinetic theory of gases, there is the empirical result known as ***Boyle's Law***. It states:

> *The volume V of a certain amount of gas confined to a container and held at a constant temperature is inversely proportional to the pressure P on the gas.*

Mathematically, this is expressed as

$$V = \frac{k}{P} \quad \text{or} \quad PV = k,$$

where k is the constant of proportionality.

Example 1.23. A Wall Street rule of thumb is that airline stock prices increase whenever petroleum stock prices go down and vice versa. Hence,

$$AP = C,$$

where A and P are the prices of the airline and petroleum stocks, respectively, and C is the constant of proportionality.

[18] The letter γ is the lower case Greek letter *gamma*.
[19] The letter ν is the lower case Greek letter *nu*.

Sometimes it must seem to the owner of a car—especially a new car—that the likelihood that a car gets a dent is inversely proportional to its age.

Finally we observe that even someone disinclined to mathematics (and thereby less fortunate) is tempted at times to invoke mathematics to make a point. Take for example this statement by David M. Knight: "The fundamental rule is that our ability to recognize the voice of God is in inverse proportion to our attachment to the things of this world." [20]

Since this book is about ordinary differential equations, the proportional relationships that we consider from now on will involve ordinary derivatives. The following examples are taken from physics and chemistry.

1.2.6 Newton's Law of Cooling

It is patently clear to anyone that an ice-cold can of soda left on a patio will eventually warm up to the outside temperature and a cup of hot chocolate set on a kitchen table will cool down to room temperature. Experimental evidence indicates that for moderate temperature differences between a body and its surroundings, the rate of change of the temperature T of the body is proportional to the difference in the temperatures of the body and its surroundings. Expressed in the language of calculus, this statement takes on the form

$$\frac{dT}{dt} \propto T - T_a$$

or

$$\frac{dT}{dt} = c(T - T_a) \tag{1.2.11}$$

where T_a is the ***ambient temperature***, or the temperature of the surroundings, and c is the constant of proportionality. This model of temperature change is known as ***Newton's law of cooling***. Clearly, the temperature of a body is decreasing when $T > T_a$ but increasing when $T < T_a$. Or, from our knowledge of calculus, the derivative dT/dt is negative when $T > T_a$ but positive when $T < T_a$. In either case, the constant c in (1.2.11) must be negative. As it is customary in the sciences to keep physical constants and parameters positive, we replace c with $-k$, where $k > 0$. Accordingly, (1.2.11) becomes

$$\frac{dT}{dt} = -k(T - T_a) \quad (k > 0). \tag{1.2.12}$$

Note that Newton's law of cooling does not result from attempting to explain the physical processes taking place, such as heat transfer between a body and its surroundings by conduction, convection, and radiation. So, in a sense all of the unexplained physical processes are represented by the constant of proportionality k. This is asking a lot of a constant and would explain the experimental evidence that Newton's law of cooling is not always valid, such as in the case of extreme temperature differences.

[20] See Knight [33, p. 80]. This unconventional example stems from Fr. Knight's tongue-in-cheek question about the importance of mathematics in God's ultimate plans for humankind.

1.2.7 Rates of Chemical Reactions

The gas ethane is a hydrocarbon: a compound consisting of only carbon and hydrogen. An ethane molecule is composed of two carbon atoms and six hydrogen atoms, so its molecular formula is C_2H_6. Ethane decomposes when it is strongly heated in the absence of air. Under certain experimental conditions, it has been observed that the rate at which this decomposition takes place is proportional to the concentration of ethane. [21] In chemistry, it is customary to denote the concentration of a compound by enclosing its formula with brackets: thus, $[C_2H_6]$ denotes the concentration of ethane. In this notation, the observation that ethane decomposes at a rate proportional to its concentration can be expressed as:

$$\text{Rate of decomposition of ethane} = k[C_2H_6],$$

where the constant of proportionality k is called a ***reaction rate constant***. It is a parameter, positive in value, which must be determined experimentally. Since ethane decomposes, the rate of change in its concentration is a negative quantity—thus, the derivative $d[C_2H_6]/dt$ is equal to the negative of the rate of decomposition:

$$\frac{d\,[C_2H_6]}{dt} = -k\,[C_2H_6]\,. \tag{1.2.13}$$

Since the mathematical manipulation of bracketed formulas is cumbersome when solving differential equations involving them, it is convenient to replace them with lower case letters. For example, by letting $x(t) = [C_2H_6]$ denote the concentration of ethane at time t, equation (1.2.13) simplifies to

$$\frac{dx}{dt} = -kx.$$

As for a second example, we consider one of several chemical reactions that have been conjectured by some scientists to take place in the earth's stratosphere to explain the alleged depletion of the ozone layer in the polar regions. [22] In this reaction, an oxygen atom (O) collides with an ozone molecule (O_3) to produce two oxygen molecules (O_2):

$$O + O_3 \to 2O_2.$$

It is believed that the rate at which the concentration of oxygen molecules increases is proportional to the product of the concentrations of oxygen atoms and ozone molecules. Translating this statement in the language of mathematics, we obtain the differential equation:

$$\frac{d[O_2]}{dt} = K[O][O_3],$$

where K is the constant of proportionality. Since the derivative is positive, K must be positive— of course, its value can only be determined experimentally.

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[21] See Atkins [3, p. 131].

[22] See Atkins [3, p. 140].

1.2.8 Newton's Second Law of Motion

Important applications of first- and second-order differential equations are found in the branch of physics called ***classical mechanics***, which is the study of the motion of bodies. A ***body*** is that which possesses both mass and extent. [23] Classical mechanics is based on three famous laws of motion formulated by Sir Isaac Newton (1642-1747); so it is also called ***Newtonian mechanics***. A proper study of the laws of motion rightfully belongs to a physics course dealing with classical mechanics. Nevertheless, the second of these laws, known as *Newton's second law of motion*, is a virtual treasure-trove of differential equations problems that are interesting, yet not too difficult. For this reason, we will use the second law of motion in the future as a source for examples to aid in making differential equations understandable and to demonstrate their usefulness in portraying physical phenomenon in mathematical terms. We discuss the second law of motion briefly here for those of you who are either not familiar or yet comfortable with it. Newton's second law of motion is the fundamental law of classical mechanics. It may be only a slight exaggeration to say that a physics or engineering mechanics course is essentially a study of how to apply this equation to a variety of situations.

A body accelerates when external forces act on it, provided they do not cancel each other out. We can add these forces, but we have to take into account that they may act in different directions. A directed quantity, of which a force is an example, [24] that is characterized by both direction and magnitude is called a ***vector***.[25] The (vector) addition of two or more vectors results in a single vector, called the ***vector sum***, that is equivalent to all the other vectors acting concurrently. The vector sum of all the forces acting on a body is called the ***resultant force*** or, simply, the ***resultant***. It is a single force equivalent to all the other external forces acting on a body in that it causes exactly the same motion in a body as do the other forces combined.

Newton's second law of motion is an equation relating the mass of a body, the resultant force acting on the body, and its acceleration. It is derived from the following three experimental observations:

1. A body accelerates in the direction of the resultant force.

2. The magnitude of the acceleration of a body of constant mass is proportional to the magnitude of the resultant force.

3. For a constant resultant force, the magnitude of the acceleration is inversely proportional to the mass of the body.

Letting m denote the mass of the body, $\boldsymbol{a}$ its acceleration, and $\mathbf{F}_{sum}$ the resultant of all the external forces acting on the body, we can encapsulate the above observations in the proportionality statement:

$$\mathbf{a} \propto \frac{\mathbf{F}_{sum}}{m}.$$

[23] This contrasts the term ***particle***, which is the idealized notion of a geometric point possessing mass.

[24] *Displacements* and *velocities* are other examples.

[25] Directed quantities also have to obey certain rules of combination before they can legitimately be considered vectors.

Thus, we end up with the simple vector equation

$$m\mathbf{a} = \lambda \mathbf{F}_{sum},$$

where λ is a constant of proportionality. The letters for acceleration and the force are in boldface to indicate that they are vector quantities. However, the letter m is not in boldface since mass is a ***scalar***, a quantity that has magnitude but no direction associated with it. For $\lambda = 1$, the vector equation becomes[26]

$$\mathbf{F}_{sum} = m\mathbf{a}. \tag{1.2.14}$$

In words:

> *The resultant force acting on a body is equal to the product of the mass of the body and its acceleration .*

This is Newton's second law of motion for the accelerated motion of a body subjected to external forces when the mass of the body remains constant.

Since ***acceleration*** is the rate of change of velocity, Newton's second law can be written as the differential equation

$$\mathbf{F}_{sum} = m\frac{d\mathbf{v}}{dt}, \tag{1.2.15}$$

where $\mathbf{v}$ is the ***velocity***, i.e., the rate of change of a body's position with respect to the time t. This statement of Newton's second law is only valid if the mass of a body is constant.

The ***momentum*** of a body is defined as the vector quantity $m\mathbf{v}$. If the mass of a body changes, then we have to use the following form of ***Newton's second law of motion***:

> *The resultant force acting on a body is equal to the rate of change of the momentum of the body.*

This is expressed mathematically by the vector equation

$$\mathbf{F}_{sum} = \frac{d}{dt}(m\mathbf{v}). \tag{1.2.16}$$

Observe that (1.2.16) reduces to (1.2.15) when m is constant.

For those situations in which the only forces causing motion of a body act either in one direction or in the opposite direction, such as forces acting vertically, directed either upward or downward, or forces acting horizontally, directed either to the right or to the left, a positive or negative sign is affixed to the magnitude of the force to indicate its direction. For forces acting horizontally, let's use the convention that positive forces are directed to the right whereas negative forces are directed to the left. Similarly, for forces acting vertically, we will regard upward as positive and downward as negative.

[26] Equation (1.2.14) results from setting $\lambda = 1$, which is permissible as long as we choose ***absolute units*** or ***consistent units***. One set of absolute units is the metric system that expresses m in *kilograms*, t in *seconds*, $\mathbf{a}$ in *meters per second squared*, and $\mathbf{F}_{sum}$ in *newtons*. See Osgood [44, pp. 51–52] or Becker [9, p. 25].

Then we merely have to add forces algebraically to obtain the resultant force. As a result, we can write (1.2.14) without using boldface letters:

$$F_{sum} = ma. \tag{1.2.17}$$

Newton's second law is used below in Example 1.24 to model the vertical motion of a body of constant mass near the earth's surface. Every body in the universe is subject to the gravitational attractive force of earth.[27] The force that the earth exerts on a body is called the ***force on the body due to gravity***. When two bodies of constant mass are released successively from the same point in space above ground level, each of them falls with approximately the same acceleration, providing the opposing resistive force from the motion of the bodies through the air is negligible in comparison with the force due to gravity. This constant acceleration is called the ***acceleration due to gravity***.[28] In this chapter, we assume that the resistive force of air has a negligible effect on the motion of a body.[29] Perhaps you have seen the rather striking demonstration of a feather falling as rapidly as a steel ball bearing inside a long, empty glass cylinder from which most of the air has been pumped out with a vacuum pump. However, when both bodies are removed from the confines of the glass cylinder, it is quite a different story. The air resistance on the feather retards considerably its motion, on the other hand, its effect on the ball bearing is barely noticeable.

The acceleration of a body of constant mass actually changes as its distance from the earth's surface varies. However, if the total distance traveled by the body during the course of its fall is not too great, we may regard its acceleration as nearly constant throughout the fall. In other words, close to the earth's surface the acceleration due to gravity is practically constant. Its magnitude is denoted by the letter g. At sea level and mid-latitudes, g is approximately 32 ft/sec^2 in the ***U. S. Customary System***.[30] In SI units,[31] short for the ***International System of Units*** and commonly referred to as the ***metric system***, g is approximately 9.8 m/s^2.

Setting $a = -g$ in (1.2.17), we obtain the force on a body of mass m due to gravity. This is what is meant by the ***weight*** of a body. In other words, the weight W of a body of mass m is

$$W = -mg,$$

where the minus sign expresses that W is directed downward, i.e., toward the center of the earth. Since weight is a force, it is expressed in force units: *pounds* in the U.S. Customary System or *newtons* in the SI system. For example, in the U.S. Customary System the unit of mass is called the *slug*. So if the mass of a body is 1 slug, then its weight is

$$W = -(1\,\text{slug}) \times (32\,\text{ft/sec}^2) = -32\,\text{pounds}.$$

Weight, being a force, is a vector quantity; however, in everyday usage, only the magnitude of the weight is stated as in saying that a bag of sugar weighs 5 pounds.

[27] In fact, according to Newton's law of universal gravitation every body in the universe exerts a gravitational force on every other body in the universe. This includes everybody too!

[28] The earth's rotation also contributes slightly to the acceleration of a falling body.

[29] We consider resistive forces such as air resistance in Chapter 3.

[30] The U.S. Customary System is the American version of the British Imperial System. Both are commonly referred to as the English or British system of units.

[31] The letters SI are from the official French name "Système International d'Unités."

Example 1.24. Model the vertical motion of a body of constant mass, such as a baseball thrown straight up or a rock released from the top of a cliff, by only considering the gravitational force that the earth exerts on the body.

Solution. In reality, there are also other forces acting on the body besides the gravitational force; but for the sake of simplicity, let us assume that their influence on the motion of the body is negligible. Let $y(t)$ denote the vertical position or height of the body directly above some fixed point on the ground at time t. Then its vertical velocity v is

$$v = \frac{dy}{dt}.$$

Even though v may look like a scalar quantity, [32] it is really a vector quantity because the derivative dy/dt can be either positive or negative. If $dy/dt > 0$, then calculus tells us y is increasing, which means the body is moving upward. In other words, a positive v means the direction of the velocity is upward. On the other hand, a negative v means the velocity is downward: the body is falling and the height y is decreasing. Implicitly we have set up a one-dimensional coordinate system; namely, the y-axis whose origin is at ground level and where all the action takes place above the ground along the positive y-axis. Moreover, positive vectors point upward and negative vectors point downward.

Since we are assuming that the only force of consequence acting on the body is the gravitational force, the acceleration of the body is $a = -g$. Thus, the motion of the body is governed by the differential equation

$$\frac{dv}{dt} = -g. \tag{1.2.18}$$

Integrating (1.2.18) with respect to t, we obtain

$$v(t) = \int -g\,dt = -gt + C.$$

Setting $t = 0$, we have

$$v(0) = -g \cdot 0 + C.$$

In other words, the constant of integration C is equal to velocity $v(0)$, i.e., the velocity of the body when it is released or thrown upward or downward. This velocity is usually denoted by the symbol v_0 and is called the ***initial velocity***. Therefore,

$$v(t) = v_0 - gt.$$

Replacing v with dy/dt, we obtain yet another differential equation:

$$\frac{dy}{dt} = v_0 - gt.$$

Integrating with respect to t again, we get

$$y(t) = \int (v_0 - gt)\,dt = v_0 t - \frac{1}{2}gt^2 + K.$$

[32] The absolute value or magnitude of the velocity is a scalar quantity called the ***speed***.

According to this formula, the height of the body at $t = 0$ is K. Thus,

$$K = y_0,$$

where $y_0 = y(0)$ is the initial height of the body. Therefore, if the resistance of the air on a body is negligible, its height at time t is

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2,$$

where y_0 and v_0 are its initial height and velocity, respectively.

Problems

Heigh-ho, Heigh-ho,
It's off to work we go.
(Whistle)
Heigh-ho, Heigh-ho, Heigh-ho,
Heigh-ho, Heigh-ho,
It's off to work we go . . .

Heigh-Ho (song in the 1937 Walt Disney movie Snow White and the Seven Dwarfs), lyrics by Larry Morey and music by Frank Churchill

What are Differential Equations?

In Problems 1 through 8, an ordinary differential equation is given. Determine the name of the independent variable, the dependent variable, and the parameter (or parameters if there is more than one). Also, give the order of the equation.

1. $\dot{N} = rN\left(1 - \frac{N}{k}\right)$
2. $\dfrac{d^2p}{ds^2} + \lambda\left(p^3 - 1\right)\dfrac{dp}{ds} = -10\sqrt{s^4 - 2}$
3. $\dfrac{d^4x}{dt^4} - a\dfrac{dx}{dt} + \dfrac{b}{5}x^5 = -10\sin^7(\gamma t)$
4. $\dfrac{d^2y}{dx^2} = \dfrac{C}{L}\sqrt{\left(\dfrac{AC}{L}\right)^2 + \left(\dfrac{dy}{dx}\right)^2}$
5. $EIy^{(4)} + py'' + ky = c(1 - x)$
6. $\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0$
7. $\dfrac{\hbar^2}{2m} \cdot \dfrac{d^2\psi}{dx^2} + \left(E - \dfrac{1}{2}kx^2\right)\psi = 0$
8. $6y''' + \eta(y'')^4 - 2\rho\omega xyy' = \cos(xy^5)$

Partial Derivatives

In Problems 9 through 12, find the first-order partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$

9. $f(x, y) = 15x^2 - 3x^4y^3 + \frac{2}{3}y^5$
10. $f(x, y) = \dfrac{x^2y}{x + 4y}$
11. $f(x, y) = x + 5e^{2x}\sin(xy)$
12. $f(x, y) = 2xy^2 - e^{4y}\ln x$

Alleged Solutions of ODEs

In Problems 13 through 18, an ordinary differential equation is given along with a function alleged to be its solution. Determine whether the alleged solution is truly a solution by means of direct substitution of the function and its derivative(s) into the equation.

13. $\dfrac{dy}{dx} = xy, \quad y = 4e^{x^2/2}$
14. $x^2y' = y^2, \quad y = \dfrac{x}{1 + Cx}$
15. $\dfrac{dy}{dx} = \dfrac{x}{y}, \quad y = -\sqrt{4 - x^2}$
16. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2},$
    $y = x\tan(\ln x), \text{ where } x > 0$
17. $2y'' - 7y' + 3y = 0, \quad y = e^{2x}$
18. $\dfrac{d^2y}{dx^2} + 16y = e^{3x}, \quad y = \dfrac{1}{25}e^{3x}$

Solutions of Basic Differential Equations

In Problems 19 through 42, find all solutions of each equation. (All integrals can be worked out using the standard integral formulas in Table 1.4 and the integration techniques that were reviewed in Section 1.2.4.)

19. $\dfrac{dy}{dx} = \cos(3x)$
20. $\dfrac{dy}{dx} = \sin\left(\dfrac{x}{5}\right)$

21. $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 5}$

22. $\dfrac{dy}{dx} = \dfrac{\ln x^2}{x}$

23. $\dfrac{dy}{dx} = \tan 5x$

24. $\dfrac{dy}{dx} = \sec^2(7x)$

25. $\dfrac{dy}{dx} = xe^{2x}$

26. $\dfrac{dy}{dx} = x \sin(3x)$

27. $\dfrac{dy}{dx} = x^2 \sin(2x)$

28. $\dfrac{dy}{dx} = e^{2x} \sin\left(\dfrac{x}{3}\right)$

29. $\dfrac{dy}{dx} = \dfrac{3}{x^2 + 4x}$

30. $\dfrac{dy}{dx} = \dfrac{x}{x^2 - 5x + 6}$

31. $\dfrac{dy}{dx} = \dfrac{4x - 10}{x^2 - 5x + 6}$

32. $\dfrac{dy}{dx} = \dfrac{1}{x^3 + 9x}$

33. $\dfrac{dy}{dx} = \dfrac{3}{x^2 + 4x + 5}$

34. $\dfrac{dy}{dx} = \sec(3x)\tan(3x)$

35. $\dfrac{dy}{dx} = \dfrac{\sin(2x)}{\cos^3(2x)}$

36. $\dfrac{dy}{dx} = 5x^2\sqrt{1 + 4x^3}$

37. $\dfrac{dy}{dx} = \dfrac{5}{\sqrt{4 - x^2}}$

38. $\dfrac{dy}{dx} = \dfrac{x^2 - 5}{x^3 - 3x^2 + 4x - 12}$

39. $\dfrac{\partial u}{\partial x} = 6x^2 y + \dfrac{2}{y}$

40. $\dfrac{\partial u}{\partial y} = 6x^2 y + \dfrac{2}{y}$

41. $\dfrac{\partial z}{\partial y} = 5 + \dfrac{x}{4 + y^2}$

42. $\dfrac{\partial z}{\partial x} = 2y - 5y^3 \csc^2 x$

ODEs from Rates of Change

In Problems 43 through 48, translate the given verbal statement into a differential equation. Use appropriate mathematical notation. Identify every letter (variable or constant) used in the equation.

43. Water is leaking out of a city swimming pool at the rate of 25 gallons per hour.

44. Let $f(v)$ be the fuel efficiency in mpg (miles per gallon) when a car is traveling at a speed of v mph (miles per hour). When the speed is 70 mph, the fuel efficiency of the car is decreasing by 0.30 mpg per mph.

45. A sewage treatment tank contains 10,000 gallons of polluted water. The tank removes 5 percent of the pollutants in the water per minute.

46. The index of refraction of a substance (such as water, flint glass, acetone, etc.) is the ratio of the speed of light in a vacuum to the speed of light in the substance. Use the variables t and s, where s denotes the distance that the light has traveled through the substance at time t.

47. A patient recovering from surgery is fed glucose intravenously at the rate of b milligrams per minute, where b is a constant.

48. For a given drop in pressure along a cylindrical pipe, the volumetric rate of flow of natural gas through the pipe is 4.5 times the fourth power of the radius of the pipe.

ODEs from Proportions

In Problems 49 through 62, translate the given verbal statement into a differential equation using appropriate mathematical notation. Write the equation in a form so that the constant of proportionality is positive. Identify every letter (variable or constant) that appears in the equation.

49. The number of squirrels in a forest preserve increases at a rate proportional to their number.

50. A basic electrical circuit that is usually considered in elementary physics courses consists of a switch that can be opened or closed, a battery, a resistor, and a capacitor connected in series. When the switch is closed,

an electrical current begins flowing through the circuit; however, it immediately begins to decrease to zero. The rate of change of the current at a given moment is proportional to its value at that moment.

51. The rate at which the volume of a melting snowball changes with time is proportional to its surface area. Given that the volume of a snowball of radius r is $4\pi r^3/3$ and its surface area is $4\pi r^2$, find the differential equation that expresses the rate at which the radius changes with time.

52. A rancher is tracking a wolf headed straight toward a snow-covered mountain. The speed with which he is able to pursue the wolf is inversely proportional to the depth of the snow.

53. There are 1,000 people in Mayberry. Whenever a rumor is started by the town's gossip, the time rate of change of the number of people who have heard the rumor is proportional to the number of people who have not yet heard the rumor. (Write the differential equation describing this situation in terms of two variables.)

54. An epidemic of rubella (German measles) breaks out in a remote, mountainous region in Argentina. Assume that P people live in this region, that this number does not change during the course of the epidemic, and that the time rate of change of the number of people infected with rubella is proportional to the product of the number who are infected and the number who are not. (Write the differential equation for the time rate of change of the number of infected people in terms of two variables.)

55. As a spherical raindrop evaporates, its volume changes at a rate proportional to its surface area. (Write the equation so that it involves only the variables volume and time.)

56. The tank of a certain toilet has a constant cross-sectional area. After the toilet is flushed and all of the water has drained from the tank, water flows from the filler tube into the tank at the rate of 3 liters per minute. However, because of a defective valve seat, waters leaks from the tank at a rate that is proportional to the square root of the depth of water in the tank. (Use this information to write a differential equation for the volume of water in the tank as it is filling. Write the equation so that the only variables that explicitly appear in it are "volume" and "time". Define every letter, whether it is a variable or a constant, that you use to come up with this equation.)

57. A tank with a constant cross-sectional area is filled with water; however, the water leaks through a small hole in its bottom.

    (a) According to ***Torricelli's law***[33] of fluid flow, the speed of the water exiting from the hole is proportional to the square root of the depth of the water in the tank.

    (b) An alternate form of Torricelli's law states that the rate with which the depth of the water in the tank decreases is proportional to the square root of the depth.

    (c) Before Torricelli formulated his law, it was thought that the depth of the water in a leaking tank would decrease at a rate proportional to the depth in the tank.[34]

58. In Aristotelian physics, objects of different weights fall at different speeds. It was believed that an object falls at a speed proportional to its weight.

59. Newton's law of universal gravitation implies that the acceleration of a body caused by the earth's gravitational pull is directed toward the center of the earth and its magnitude is inversely proportional to the square of the distance between the body and the center of the earth. Assuming all other external forces are negligible, use this infor-

[33] Besides formulating this law of fluid mechanics, Evangelista Torricelli (1608–1647), an Italian mathematician and physicist, is also remembered for inventing the mercury barometer in 1643. In Florence, he served briefly as Galileo Galilei's assistant and secretary. He inherited Galileo's appointment, after the latter's death in 1642, as philosopher and chief mathematician to the court of Grand Duke Ferdinando II of Tuscany.

[34] See Driver [21, p. 454].

mation to find a first-order differential equation relating the variables, distance and velocity, of a freely falling body.

60. A body is falling to earth. Apart from the force due to gravity, assume that the only other non-negligible force acting on the body is the drag force. The force due to gravity points downward. In the U. S. Customary System, it has a magnitude of $mg = 32m$, where m is the mass of the body. The drag force results from the force exerted by the air on the body opposing its motion—and so points upward. Assume that its magnitude is proportional to the velocity of the body. The product of the mass of the body and the rate of change in its velocity is equal to the sum of the drag force and the force due to gravity.

61. The rate with which water (H_2O) forms in the reaction

$$OH + H_2 \rightarrow H_2O + H,$$

is proportional to the product of the concentrations of hydrogen molecules (H_2) and hydroxide ions (OH).

62. One reaction among several reactions that may take place in the decomposition of ozone (O_3) in the stratosphere is

$$O_3 \rightarrow O_2 + O.$$

The rate at which the concentration of ozone is decreasing at any instant is proportional to its concentration at that very instant.

Newton's Second Law of Motion

63. A body with mass m is moving along the positive x-axis due to a force that attracts it to the origin with a magnitude proportional to its distance from the origin. The frictional force opposing the motion of the body is proportional to the body's weight. Use Newton's second law of motion to find the second-order differential equation governing the motion of the body along the x-axis.

64. Experiments confirm that a good model for the drag force on a parachutist is a force that is proportional to the square of the velocity of the parachutist. The weight of the parachutist is mg, where g is the acceleration due to gravity and m is the total mass of the parachutist, including the parachute and other equipment. Other than the weight and drag force, ignore all other forces acting on the parachutist. Find the rate of change of the velocity of the parachutist using Newton's second law of motion.

65. Suppose that it were possible to drill a straight tunnel from Memphis to the diametrically opposite point on the other side of the earth. Imagine dropping a bowling ball into the tunnel. If, for the sake of simplicity, we assume that the earth is a homogeneous sphere, then it can be shown using physics, trigonometry, and calculus that the gravitational force exerted on the ball by the earth is directly proportional to the distance of the ball from the center of the earth. Ignore all other forces exerted on the ball.

    (a) Translate the above information concerning the gravitational force exerted on the bowling ball into a mathematical formula. Express it in terms of the bowling ball's position from the center of the earth.

    (b) Use Newton's second law to obtain the differential equation that models the motion of the bowling ball.

66. (a) The Greek philosopher Aristotle (384–322 B. C.) taught that an object falls at a speed proportional to its weight. This was the prevailing view of university professors, even as late as the $17th$ century. To demonstrate the falsity of this notion, Galileo,[35] as legend has it, dragged cannonballs and musket balls of different weights, up the spiral staircase of the Leaning Tower of Pisa and dropped them from the top story, a height of approximately 180 feet. With the aid of Newton's second law of motion, estimate

[35] A narrative of the life of Galileo Galilei (1564–1642), Italian astronomer and physicist extraordinaire, is wonderfully told in the book *Galileo's Daughter* by Dava Sobel [50]. Letters written to him by his eldest daughter, Suor Maria Celeste, a cloistered nun of the Order of the Poor Clares, are woven masterfully into the story of the life of this incredible genius.

the time it takes for a 10-pound cannonball to fall to the ground, assuming that air resistance has a negligible effect on the ball's motion.

(b) According to Aristotelian physics, how long would it take a one-pound musket ball to reach the ground? Base your answer on part (a) and compare it with the time predicted by Newtonian physics.

67. A baseball is thrown vertically upward. The thrower's hand is y_0 feet from the ground when the ball is released. The baseball attains a height of h feet t_1 seconds after its release. After reaching a maximum height, the baseball then falls back to this point at some time $t_2 > t_1$.

    (a) Show that the velocity of the baseball at the moment of release is the product of g and the average of t_1 and t_2.

    (b) Show that $h = y_0 + \frac{1}{2} g t_1 t_2$.

Lengths of Plane Curves

67. Find a general formula for the length of a plane curve defined by the parametric equations (1.2.1) by using equation (1.2.2).

68. Use the formula found in Problem 67 to find the circumference of a circle given by the parametric equations:

$$x = r \cos t, \ y = r \sin t.$$

69. Use the result of Problem 67 to find the general formula for the length of a curve defined by the function $y = f(x)$ for $\alpha \le x \le \beta$. *Hint.* Set $x = t$ so that $y = f(t)$.

Stopping Distance of Trains

70. A 150-car freight train is approaching Carbondale, Illinois at a constant speed of 50 feet per second (approximately 34 mph). As the train nears a railroad crossing, the locomotive engineer sees a car stalled on the tracks. It takes him 4 seconds to react before he applies the brakes. It then takes the train another 1.25 minutes to come to a full stop. Assume that the deceleration of the train is constant while the brakes are being applied.

    (a) What is the deceleration of the train?

    (b) How far does the train travel from the moment the engineer sees the car until the time the train comes to a full stop? Do an Internet search to compare your answer to actual stopping distances of trains.

Poiseuille's Law

71. The volumetric rate of flow of a fluid, such as water or natural gas, through a pipe of circular cross section is determined by a number of variables: the fluid's viscosity, the pipe's radius and length, and the difference in pressure between the ends of the pipe. In parts (a) through (d), convert the given proportional relationship into a differential equation.

    (a) The volumetric rate of flow is proportional to the difference in pressure between the ends of the pipe when all other variables are held constant.

    (b) The volumetric rate of flow is inversely proportional to the length of the pipe when all other variables are held constant.

    (c) The volumetric rate of flow is inversely proportional to the fluid's viscosity when all other variables are held constant.

    (d) The volumetric rate of flow is proportional to the pipe's radius to the fourth power when all other variables are held constant

    (e) Combine the proportional relationships given in parts (a) through (d) into one differential equation. The resulting equation is known as ***Poiseuille's law***.

    (f) Explain why the air ducts for ventilating buildings generally have a large radius.

    (g) If arteriosclerosis reduces the effective radius of a person's artery by 10%, by what factor is the volumetric rate of flow of the blood through the artery reduced?

Ptolemaic and Copernican Models of the Universe

72. The Greek philosopher Aristotle (4*th* century B. C.) taught that the Sun and other heavenly bodies revolved around the Earth but that Earth itself was immobile, fixed in position at the center of the universe. In the 2*nd* century A. D., Claudius Ptolemy, an Alexandrian astronomer, refined Aristotle's view by theorizing that not only did the planets move around the Earth in circular orbits called ***deferents***; but each planet also moved in a second smaller circular orbit, called an ***epicycle***, the center of which moved along the planet's deferent. This model, known as the ***Ptolemaic system***, helped explain the observed retrograde motions of the planets, something Aristotle's thesis failed to do, and was the prevailing cosmologic theory as late as the seventeenth century. Since the Sun moved around the Earth in the Ptolemaic system, the daily transition of night to day and back again to night meant that the Sun had to make one complete revolution around the Earth every twenty-four hours. This implied the Sun moved at an enormous speed; and if the movements of even more distant stars in the heavens were to be explained, they had to move at even greater speeds. This became one of many arguments against the correctness of the Ptolemaic system.

    (a) In accordance with the Ptolemaic system, compute the speed of the Sun about the Earth given a mean distance of some $93,000,000$ miles between them.

    (b) Research the Copernican system, the heliocentric theory proposed by Nicolaus Copernicus (1473–1543, a Polish astronomer and cleric, and contrast it with the Ptolemaic system.

Computer Algebra System Problems

73. Find the solutions of the differential equations in Problems 19 through 42 with the aid of a CAS. Compare these solutions with the solutions that you obtained by just using pencil and paper. For some problems, the CAS may seem to give different answers; for these cases, provide the necessary algebra to show the answers are equivalent.

74. The rate of change of a quantity Q with respect to the time t is equal to

$$2\pi\sqrt{t}e^{-t} - 5t^4 \ln\sqrt{3+t^4}.$$

Use a CAS to determine Q up to a constant.

Chapter 2

Separable Equations

To be, or not to be–that is the question . . .

Hamlet's soliloquy in Act 3, Scene 1, *The Tragedy of Hamlet*
by William Shakespeare

To separate, or not to separate–that is the question . . .

musings of a student faced with solving
a differential equation

2.1 Introduction to Separable Equations

In some of the problems in Chapter 1, we were given a differential equation and a so-called alleged solution and then had to determine if this alleged solution truly satisfied the equation. In real-life applications, however, this is not what happens. The differential equation comes without the solution. It is we who have to somehow come up with the solution. Consequently, it now becomes incumbent on us to learn how to solve the kinds of equations that we are most likely to encounter: in future courses and eventually during the course of our professional careers. The most elementary kind of differential equations in engineering and the mathematical sciences are called *first-order separable equations*. So just exactly what is a separable equation?

Separable Equation

Definition 2.1. A first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.1.1}$$

is said to be ***separable*** if

$$f(x, y) = g(x) \cdot h(y), \tag{2.1.2}$$

where g depends only on x and h depends only on y.