Math 231 – Exam 1 Name Key

Friday, September 21, 2007 Section ______

Show your work; otherwise you may lose points if your explanation is inadequate. Circle your answers. Calculators are allowed on this exam.

Part 1. (10 pts) Fill in the blanks with the correct answers.

(a) The only analytical method for solving ordinary differential equations that we have studied up to this point in the course is known as separation of variables (2 points will be deducted for any misspellings)

(b) Newton's second law of motion for a body of constant mass m expressed as an equation is $F = ma$, where a and F are parameters representing the acceleration and resultant force, respectively.

Part 2. (10 pts)

Someone claims that $xy^2 + 4y = e^y$ is an implicit solution of the differential equation $y' = \dfrac{2xy+4}{e^y - y^2}$. Is that claim correct? If not, what differential equation does it solve?

Differentiating both sides of $xy^2+4y=e^y$ with respect to x, we obtain

$$\frac{d}{dx}(xy^2+4y) = \frac{d}{dx}e^y \Rightarrow x(2yy') + y^2 + 4y' = e^y y'$$

$$\Rightarrow y'(2xy+4-e^y) = -y^2$$

$$\Rightarrow y' = \frac{-y^2}{2xy+4-e^y}$$

Conclusion. The claim is incorrect. $xy^2+4y=e^y$ is an implicit solution of $y' = \dfrac{y^2}{e^y-2xy-4}$

Part 3. (40 pts) Solve each of the following equations. Find explicit solutions when it is possible to do so.

(a) $2x\dfrac{dy}{dx} + \dfrac{3}{xy} = 0$ $\Rightarrow 2xy\dfrac{dy}{dx} = -\dfrac{3}{x} \Rightarrow 2y\,dy = -\dfrac{3}{x^2}dx$.

Integrating, we get

$$\int 2y\,dy = -3\int x^{-2}dx + C \Rightarrow y^2 = -3(-x^{-1}) + C$$

$$\Rightarrow y^2 = \frac{3}{x} + C$$

$$\Rightarrow y = \pm\sqrt{C + \frac{3}{x}}$$

or $y = \pm\sqrt{\dfrac{Cx+3}{x}}$

(b) $\dfrac{\partial z}{\partial x} = 5 + \sin\left(\dfrac{x}{y}\right)$

Integrating wrt x, we have

$$z(x,y) = \int \left(5 + \sin\left(\frac{x}{y}\right)\right) dx + f(y) = \int 5\,dx + \int \sin\left(\frac{x}{y}\right) dx + f(y)$$

Hence, $z(x,y) = 5x - y\cos\left(\dfrac{x}{y}\right) + f(y)$

(c) $\dfrac{dy}{dx} = \dfrac{1}{x^2 - x}$

find the partial fraction expansion of $\dfrac{1}{x^2-x}$.

$y = \int \dfrac{1}{x^2-x}\,dx$. Now, $\dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$\Rightarrow A(x-1) + Bx = 1$. $\quad x = 0 \Rightarrow -A = 1 \Rightarrow A = -1$

$\quad x = 1 \Rightarrow B = 1$

Then, $y = -\int \dfrac{dx}{x} + \int \dfrac{dx}{x-1} + C \Rightarrow y = -\ln|x| + \ln|x-1| + C$

or $y = \ln\left|\dfrac{x-1}{x}\right| + C$

(d) $\dfrac{dx}{dt} = \dfrac{3x - 2tx}{\sqrt{4-t^2}}$

Separating variables, we have $\dfrac{dx}{dt} = \dfrac{x(3-2t)}{\sqrt{4-t^2}} \Rightarrow \dfrac{1}{x}\dfrac{dx}{dt} = \dfrac{3-2t}{\sqrt{4-t^2}}$

$\Rightarrow \dfrac{1}{x}\dfrac{dx}{dt} = \dfrac{3}{\sqrt{4-t^2}} - \dfrac{2t}{\sqrt{4-t^2}}$. Integrating, we obtain

$$\int \frac{1}{x}\,dx = \int \frac{3}{\sqrt{4-t^2}}\,dt - \int \frac{2t}{\sqrt{4-t^2}}\,dt + K$$

$$\Rightarrow \ln|x| = \frac{3}{2}\int \frac{dt}{\sqrt{1-\left(\frac{t}{2}\right)^2}} + \int \frac{-2t\,dt}{\sqrt{4-t^2}} + K$$

$$= 3\sin^{-1}\left(\frac{t}{2}\right) + 2\sqrt{4-t^2} + K.$$

Exponentiating,

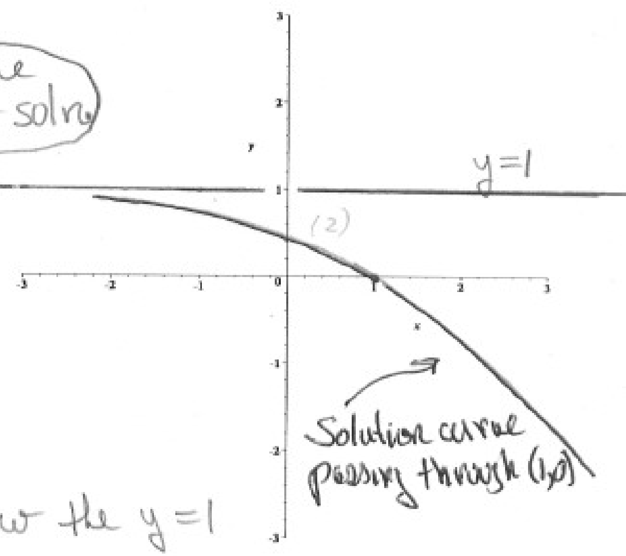
$|x| = e^{3\sin^{-1}\left(\frac{t}{2}\right) + 2\sqrt{4-t^2} + K} \Rightarrow x = Ce^{3\sin^{-1}\left(\frac{t}{2}\right) + 2\sqrt{4-t^2}}$

Part 4. (10 points) Consider $\frac{dy}{dx}=(y-1)^3$

(a) Find constant solutions, if any, and write their equations here.

(2) Set $\frac{dy}{dx}=0$: $(y-1)^3=0 \Rightarrow$ $y(x)\equiv 1$ is the only constant soln

(b) Compute y'' and circle your answer. Then sketch the graph of the solution that passes through the point $(1,0)$ using the first and second derivative tests.



$$y'' = \frac{d}{dx}(y-1)^3 = 3(y-1)^2\frac{d}{dx}(y-1)$$
$$= 3(y-1)^2 y' = 3(y-1)^2(y-1)^3$$

(2) $\boxed{\therefore\ y'' = 3(y-1)^5}$

(1) For $y<1$, $y'<0 \Rightarrow$ Solution curves below the $y=1$ line are falling.

And as $y''<0$ for $y<1$, solution curves below the $y=1$ line are concave down.

(c) Find the equation of the line that is tangent to the solution curve in part (b) at the point $(1,0)$.

The slope of the line is $\left.\frac{dy}{dx}\right|_{(1,0)} = (0-1)^3 = -1$.

(3) By the point-slope form of the eqn of a line, $y-0=-1(x-1)$

$\Rightarrow$ $y = 1-x$

Part 5. (10 points)

Find the explicit solution of the initial value problem: $\frac{dx}{dt}=\frac{2t}{x^2}$, $x(3)=-2$.

Separating variables, we have $x^2\,dx = 2t\,dt$.

Integrating + using the initial condition, we get

$$\int_{-2}^{x} u^2\,du = \int_{3}^{t} 2v\,dv \Rightarrow \left.\frac{u^3}{3}\right|_{-2}^{x} = \left.v^2\right|_{3}^{t}$$

$$\Rightarrow \frac{x^3}{3} - \frac{(-2)^3}{3} = t^2 - 9$$

$$\Rightarrow x^3 - (-8) = 3t^2 - 27$$

$$\Rightarrow x^3 + 8 = 3t^2 - 27 \Rightarrow x^3 = 3t^2 - 35$$

$$\Rightarrow x(t) = \sqrt[3]{3t^2-35}$$

Part 6. (20 pts)

(a) Foxes are introduced in a forest preserve in order to reduce the rabbit population. The result is that the number of rabbits decreases at a rate inversely proportional to their number. Translate this into differential equation (but do not solve it). Identify every variable and constant that you use in the equation. Use the convention that constants of proportionality are positive.

Let t be the time.

Let $k>0$ be the constant of proportionality (3)

Let $p(t)$ be the rabbit population at time t.

The 2nd sentence says

$$\frac{dp}{dt} = -\frac{k}{p}.$$

(7)

(b) A boy is rowing a boat in still water. The boat is moving northward at a uniform velocity of 2 miles per hour. Assume that the resistive force of the water on the boat is proportional to the velocity of the boat. Determine the differential equation modeling the velocity of the boat after the boy ships the oars (i.e., places the oar in a resting position in the boat). Translate this into differential equation (but do not solve it). Identify every variable and constant that you use in the equation. Use the convention that constants of proportionality are positive.

Let $v(t)$ be the velocity of the boat at time t where $t=0$ is when the boy ships the oars.

For $t \geq 0$, the only horizontal force acting on the boat is the resistive force F_r, where

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$$F_r = -kv \quad (k>0)$$

where k is the constant of proportionality.

Applying Newton's second law of motion,

$$ma = F_r$$

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$$\Rightarrow m\frac{dv}{dt} = -kv \quad \text{or} \quad \frac{dv}{dt} = -\frac{k}{m}v$$

where m is combined mass of the boat and boy.