Integration by Parts

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Introduction. Every student taking a course in calculus in his freshman or sophomore year learns the technique of evaluating an integral by the method known as "*integration by parts*" as embodied in the following well-known rule:

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx.$$

The above rule is also given in the form

$$\int u\, dv = uv - \int v\, du.$$

Students often find this rule puzzling and have trouble in picking the right u and v. I have adopted the following procedure, which may not be new, but which appears to have succeeded well. I will illustrate the procedure with five examples.

Example 1.

$$\text{Evaluate } \int x^2 \sin x\, dx.$$

The student should factor the integrand into a product of two functions, one whose successive derivatives eventually become zero, and the other whose successive anti-derivatives are easily obtained. In our example, $f(x) = x^2$ can be differentiated successively and the third derivative is zero. The function, $g(x) = \sin x$, when integrated successively, gives $-\cos x$, $-\sin x$, and $\cos x$, respectively. These successive derivatives of $f(x)$ and anti-derivatives of $g(x)$ are written in the following format:

$f(x)$ and its successive derivatives		$g(x)$ and its successive anti-derivatives
x^2	$(+)$	$\sin x$
$2x$	$(-)$	$-\cos x$
2	$(+)$	$-\sin x$
0		$\cos x$
	$(-)$	

Then he is asked to draw arrows as indicated above. Products of functions connecting these arrows are formed and the value of the required integral is

$$\int x^2 \sin x \, dx = (+)(x^2)(-\cos x) - (2x)(-\sin x)$$
$$+ (2)(+1)(\cos x) - \int 0 \cdot \cos x \, dx.$$

The signs for these products are always $+$, $-$, $+$, $-$, etc. The products with the appropriate signs are summed except the product with the horizontal arrow which is integrated, and this clearly is zero. Thus,

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

The above method can always be used as long as the integrand can be factored as $f(x)g(x)$, where f can easily be differentiated successively and goes to zero and g can easily be integrated successively. Thus, the procedure is workable for $x^n \sin ax$, $x^n \cos ax$, or $x^n e^{ax}$.

Example 2.

$$\int e^{2x} \sin x \, dx.$$

Here successive derivatives of either e^{2x} or $\sin x$ would never become zero. Both are easy to integrate or differentiate successively and the procedure described above is modified slightly.

$f(x)$ and its successive derivatives		$g(x)$ and its successive anti-derivatives
e^{2x}	$(+)$	$\sin x$
$2e^{2x}$	$(-)$	$-\cos x$
$4e^{2x}$		$-\sin x$
	$(+)$	

Successive differentiation and integration of $f(x)$ and $g(x)$ is carried out until we reach a stage where the product of the derivative and anti-derivative at that stage is a constant multiple of the function whose integral we wish to evaluate. Proceeding as before, we obtain:

$$\int e^{2x} \cdot \sin x \, dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x \, dx$$

or

$$\int e^{2x}\sin x \, dx = -\frac{1}{5} e^{2x}\cos x + \frac{2}{5} e^{2x}\sin x + C.$$

This procedure is useful for integrating functions like $e^{ax}\sin bx$, or $e^{ax}\cos bx$.

Example 3.

Evaluate $\int x^2 \ln x \, dx$.

Although the third derivative of x^2 is zero, the procedure used in the first example is not fruitful, since successive integrals of $\ln x$ are hard to obtain. We can, of course, easily find the successive derivatives of $\ln x$, and integrate x^2 successively, but in this case we will never reach a stage where the product of the derivative and the anti-derivative at that stage is a constant multiple of $f(x)g(x)$. In such situations, we stop at the stage where we see that the product of the derivative and the integral at that stage is easily integrable.

$$\begin{array}{ccc} f(x)\text{ and its} & & g(x)\text{ and its} \\ \text{successive derivatives} & & \text{successive anti-derivatives} \\ \ln x & (+) & x^2 \\ \dfrac{1}{x} & \longleftarrow & \dfrac{x^3}{3} \\ & (-) & \end{array}$$

We see that the product of the first derivative of $\ln x$ and the anti-derivative of x^2 is $x^2/3$, which can easily be integrated. So we stop and follow our general rule:

$$\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.$$

A word of caution. When the successive derivatives of $f(x)$ do not show a zero at any stage, and the product of the derivative of f and the integral of g at some stage becomes exactly equal to $f(x)g(x)$ with a plus sign, this method does not give the value of the integral.

Example 4.

$$\int \sin^2 x \, dx.$$

f and successive derivatives of f		g and successive anti-derivatives of g
$\sin x$	$(+)$	$\sin x$
$\cos x$	$(-)$	$-\cos x$
$-\sin x$	$(+)$	$-\sin x$

We stopped with the second derivative since at this stage the product is $\sin^2 x$. Hence,

$$\begin{aligned}\int \sin^2 x \, dx &= -(\sin x)\cos x + \cos x \sin x + \int \sin^2 x \, dx \\ &= 0 + \int \sin^2 x \, dx.\end{aligned}$$

This is true, but useless, for our purpose.

$\int \sin^2 x \, dx$ can be evaluated using integration by parts if we proceed as follows:

$$\begin{aligned}\int \sin^2 x \, dx &= \int (1 - \cos^2 x) \, dx = x - \int \cos^2 x \, dx \\ &= x - \int \cos x \cdot \cos x \, dx \\ &= x - \left[\sin x \cos x - \int \sin x(-\sin x) \, dx\right] \\ &= x - \sin x \cos x - \int \sin^2 x \, dx,\end{aligned}$$

which yields

$$\int \sin^2 x \, dx = \frac{1}{2}\left[x - \sin x \cos x\right] + C.$$

Integrals of the type $\int \sin ax \cos bx \, dx$, $\int \sin ax \sin bx \, dx$, and $\int \cos ax \cos bx \, dx$ can be evaluated using this procedure when $a \neq b$ and the student need not remember any trigonometric identities in evaluating these integrals involving products of sines and cosines.

Example 5.

$$\text{Evaluate } \int \sin 5x \cos 3x \, dx = I.$$

$$\begin{array}{ccc} \sin 5x & (+) & \cos 3x \\ 5\cos 5x & (-) & +\frac{1}{3}\sin 3x \\ -25\sin 5x & \longleftarrow & -\frac{1}{9}\cos 3x \\ & (+) & \end{array}$$

$$I = \tfrac{1}{3}\sin 3x \sin 5x + \tfrac{5}{9}\cos 3x\cos 5x + \tfrac{25}{9} I$$

$$I = -\tfrac{3}{16}\sin 3x \sin 5x - \tfrac{5}{16}\cos 3x \cos 5x + C.$$

Applications of Intermediate Algebra: A Possible Alternative

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Introduction

In recent years, many four-year colleges and universities have begun to offer a course in Intermediate Algebra. There are many students who need Intermediate Algebra in order to proceed with the degree program in their major area. At Oregon State University, for example, approximately 1500 students enroll in Intermediate Algebra each year. These students enroll in the course for a variety of reasons, and their backgrounds in mathematics are very widely dispersed. Some students just need a refresher course. Other students have had little or no high school algebra and may have been away from mathematics for three or four years. There are students in Home Economics or Liberal Arts who may take Intermediate Algebra as a terminal course; students from the School of Business who need a