UNIFORMLY CONTINUOUS AND ASYMPTOTICALLY STABLE SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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1. Introduction

We investigate the asymptotic behavior of solutions of the scalar linear homogeneous Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{1.1}$$

for $t \geq 0$, where a and b are real-valued functions that are continuous on the respective domains $[0, \infty)$ and

$$\Omega := \{(t,s)\colon 0 \leq s \leq t < \infty\}.$$

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Definition 1.1.

A *solution* of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{1.1}$$

on $[0, T)$, where $0 < T \leq \infty$, with an initial value $x_0 \in \mathbb{R}$ is a continuous function $x\colon [0, T) \to \mathbb{R}$ that satisfies (1.1) on $(0, T)$ such that $x(0) = x_0$.

Since a and b are continuous functions, a solution $x(t)$ satisfying the initial condition $x(0) = x_0$ exists on the entire interval $[0, \infty)$ and is unique. (cf. [4, p. 5] or [7, pp. 23–27, p. 221]).

At times, instead of $x(t)$, we write $x(t, 0, x_0)$ for the sake of clarity.

Furthermore, for each $t_0 > 0$ and each continuous initial function $\varphi\colon [0, t_0] \to \mathbb{R}$, there is a unique continuous function $x\colon [0, \infty) \to \mathbb{R}$ that satisfies (1.1) on (t_0, ∞) such that $x(t) \equiv \varphi(t)$ on $[0, t_0]$.

(cf. [8, p. 179])

We denote this solution by $x(t, t_0, \varphi)$.

Example.

$$x'(t) = -\frac{1}{1+t}\,x(t) + \int_0^t \frac{\cos s}{(1+t)^3}\,x(s)\,ds \tag{E}$$

Initial function: $\varphi(t) = 3\sin(2t) - 2t$ on the interval $[0,2]$.

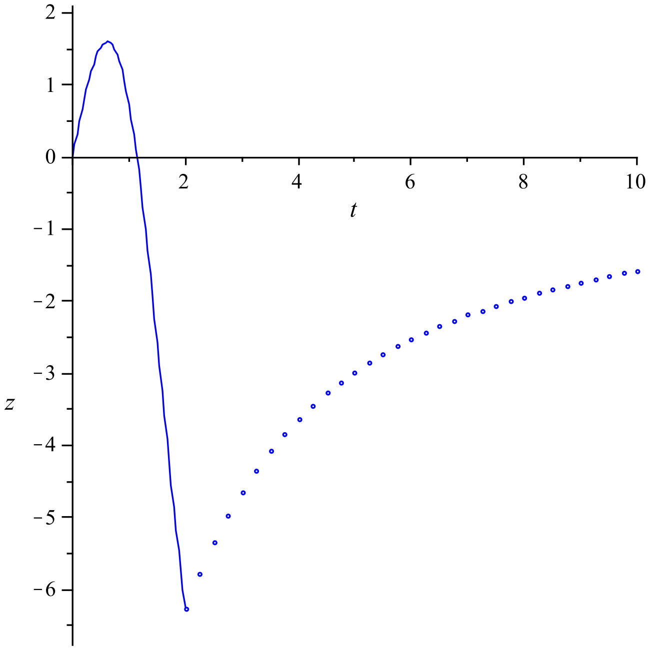

FIG. 1. A numerical solution of (E)

(from the Maple Application Center: *Scalar Volterra Integro-Differential Equations* by Becker, see "6. Integro-differential equation 2")

In fact, all solutions of (E) approach 0 as $t \to \infty$. This can be shown with the Liapunov functional

$$V(t,\psi(\cdot)) := (1+t)\psi^2(t) + \int_0^t \left[\frac{1}{1+s} - \int_s^t \frac{|\cos s|}{(1+u)^2}\,du\right]\psi^2(s)\,ds.$$

Employing Liapunov functionals that were constructed for (1.1) by Burton in [8, p. 122] and [9] and by Becker in [5, p. 34], with some modifications, we obtain a number of conditions involving a and b so that the zero solution of (1.1) is stable and its other solutions approach zero as $t \to \infty$. Typically, one looks for conditions so that the derivative $x'(t)$ is bounded on $[0, \infty)$.

However, we suggest that in investigations of stability more emphasis ought to be placed on the uniform continuity of the solutions. The reason for this derives from the observation: every differentiable function with a bounded derivative is uniformly continuous—but not conversely as attested by the function

$$f(t) = \frac{\sqrt{t}}{1+t}. \tag{1.2}$$

Even though its derivative f' is unbounded on $[0, \infty)$, f is uniformly continuous on $[0, \infty)$. Moreover, $f(t)$ tends to zero as $t \to \infty$.

A thesis then of this paper is that it takes fewer and less stringent conditions to imply solutions are uniformly continuous and approach zero than it takes to imply solutions have bounded derivatives and approach zero.

Notation.

- $C[t_0, t_1]$ (resp. $C[t_0, \infty)$) denotes the set of all continuous real-valued functions on $[t_0, t_1]$ (resp. $[t_0, \infty)$).

- For $\varphi \in C[0, t_0]$, let
$$|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \leq t \leq t_0\}.$$

- $L^1[0, \infty)$ denotes the set of all continuous real-valued functions that are absolutely Riemann integrable on $[0, \infty)$. That is, by $g \in L^1[0, \infty)$ we will mean that
$$\lim_{t \to \infty} \int_0^t |g(s)| \, ds$$
exists and is finite.

- $L^2[0, \infty)$ denotes the set of all continuous real-valued functions that are square integrable on $[0, \infty)$. That is, by $h \in L^2[0, \infty)$ we will mean that h is continuous on $[0, \infty)$ and $h^2 \in L^1[0, \infty)$.

The first theorem in this paper lists conditions ensuring that all solutions of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{1.1}$$

approach 0 as $t \to \infty$. Its proof relies on

Barbălat's Lemma. *If $f\colon [0,\infty) \to \mathbb{R}$ is both uniformly continuous and Riemann integrable on $[0,\infty)$, then $f(t) \to 0$ as $t \to \infty$.*

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I. Barbălat, Systèms d'équations différentielle d'oscillations nonlinéaires, *Rev. Roumaine Math. Pures Appl.* **4** (1959), 267–270.

Recent proof:
H. Logemann and E. P. Ryan, Asymptotic Behaviour of Nonlinear Systems, *Amer. Math. Monthly* **111** (2004), 864–889.

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The idea that Barbălat's Lemma might prove fruitful in looking for conditions so that $x(t) \to 0$ as $t \to \infty$ originated from obtaining the result

$$\int_0^t a(s)\,|x(s)|\,ds \le k, \quad \forall t \ge 0 \qquad (k \text{ a constant}) \tag{B}$$

from one of the Liapunov functionals in this paper.

Definitions of Stability

Definition 1.2.

The zero solution of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{1.1}$$

is

(1) *stable* if for every $\epsilon > 0$ and every $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$ implies that $|x(t, t_0, \varphi)| < \epsilon$ for all $t \geq t_0$.

(2) *globally asymptotically stable* (*asymptotically stable in the large*) if it is stable and if every solution of (1.1) approaches zero as $t \to \infty$.

2. A UNIFORMLY CONTINUOUS LIAPUNOV FUNCTIONAL

Lemma 2.1. *Let* $f \in C[0,\infty)$*. If* $\lim_{t\to\infty} f(t)$ *exists and is finite, then* f *is uniformly continuous on* $[0,\infty)$*.*

Consequently,

Lemma 2.2. *Let* $f \in C[0,\infty)$*. If* f *is Riemann integrable on* $[\tau,\infty)$ *for some* $\tau \geq 0$*, then* $\int_0^t f(s)\,ds$ *is uniformly continuous on* $[0,\infty)$*.*

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Theorem 2.3. *Let* $a\colon [0,\infty) \to [0,\infty)$ *and* $b\colon \Omega \to \mathbb{R}$ *be continuous functions. If*

$$\int_s^t |b(u,s)|\,du \leq a(s) \tag{2.1}$$

for all $t \geq s \geq 0$*, then the zero solution of*

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{2.2}$$

is stable. Furthermore, if for some $t_1 \geq 0$ *there is a constant* $k > 0$ *such that*

$$a(t) \geq k \tag{2.3}$$

for all $t \geq t_1$ *and a constant* $\lambda \in (0,1)$ *such that*

$$\int_s^t |b(u,s)|\,du \leq \lambda a(s) \tag{2.4}$$

for all $t \geq s \geq t_1$*, then every solution* $x(t)$ *of* (2.2) *belongs to* $L^1[0,\infty)$ *and is uniformly continuous on* $[0,\infty)$*. Moreover, the zero solution is globally asymptotically stable.*

Proof.

Part 1. [The zero solution is stable.]

Define the Liapunov functional $V\colon [0,\infty)\times C[0,\infty)\to[0,\infty)$ by

$$V(t,\psi(\cdot)) := |\psi(t)| + \int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right] |\psi(s)|\,ds. \quad (2.5)$$

By (2.1), $V(t,\psi(\cdot)) \geq |\psi(t)|$ for all $t \geq 0$.

For any $t_0 \geq 0$ and $\varphi \in C[0,t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the solution of (2.2) on $[0,\infty)$ with $x(t) = \varphi(t)$ for $0 \leq t \leq t_0$.

Now consider $V(t) := V(t, x(\cdot))$ and its derivative. Since $x(t)$ is continuously differentiable on $[t_0,\infty)$, $|x(t)|$ has a right derivative $D_r|x(t)|$ given by

$$D_r|x(t)| = \begin{cases} x'(t)\operatorname{sgn} x(t), & \text{if } x(t) \neq 0 \\ |x'(t)|, & \text{if } x(t) = 0 \end{cases} \quad (2.6)$$

for all $t \geq t_0$ (cf. [14, p. 26]).

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$$\int_s^t |b(u,s)|\,du \leq a(s) \quad \forall\, t \geq s \geq 0. \quad (2.1)$$

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds, \quad (2.2)$$

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$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds, \tag{2.2}$$

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Thus, the right derivative of V for $t \geq t_0$ is

$$\begin{aligned} D_r V(t) &= D_r|x(t)| + \frac{d}{dt}\int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right]|x(s)|\,ds \\ &\leq -a(t)|x(t)| + \int_0^t |b(t,s)||x(s)|\,ds \\ &\qquad + a(t)|x(t)| - \int_0^t |b(t,s)||x(s)|\,ds \end{aligned}$$

and so

$$D_r V(t) \leq 0. \tag{2.7}$$

Thus,

$$|x(t)| \leq V(t) \leq V(t_0) \tag{2.8}$$

for all $t \geq t_0$, where

$$\begin{aligned} V(t_0) &= |\varphi(t_0)| + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\,du\right]|\varphi(s)|\,ds \\ &\leq M(t_0)|\varphi|_{t_0} \end{aligned}$$

and

$$M(t_0) := 1 + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\,du\right]ds. \tag{2.9}$$

For a given $\epsilon > 0$, let $\delta = \epsilon/M(t_0)$. Then for $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \delta$, we have

$$|x(t)| \leq V(t_0) \leq M(t_0)|\varphi|_{t_0} < \delta M(t_0) = \epsilon \tag{2.10}$$

for all $t \geq t_0$, which proves stability.

Part 2. [Every solution belongs to $L^1[0,\infty)$.]

Suppose for some $t_1 \geq 0$, $\exists\, k > 0$ and $\lambda \in (0,1)$ such that

$$a(t) \geq k, \quad \forall\, t \geq t_1 \tag{2.3}$$

$$\int_s^t |b(u,s)|\, du \leq \lambda a(s), \quad \forall\, t \geq s \geq t_1. \tag{2.4}$$

Let $\gamma := \sqrt{\lambda}$ and

$$V_\gamma(t) := |x(t)| + \int_0^t \left[\gamma a(s) - \frac{1}{\gamma}\int_s^t |b(u,s)|\, du\right] |x(s)|\, ds. \tag{2.11}$$

By (2.4),

$$V_\gamma(t) \geq |x(t)| \tag{2.12}$$

for all $t \geq t_1$. And

$$\begin{aligned} D_r V_\gamma(t) \leq -a(t)|x(t)| &+ \int_0^t |b(t,s)||x(s)|\, ds \\ &+ \gamma a(t)|x(t)| - \frac{1}{\gamma}\int_0^t |b(t,s)||x(s)|\, ds \\ \leq -(1-\gamma)a(t)|x(t)| & \end{aligned} \tag{2.13}$$

for all $t \geq \tau$, where $\tau := \max\{t_0, t_1\}$. Then, because of (2.3),

$$D_r V(t) \leq -k(1-\gamma)|x(t)|. \tag{2.14}$$

An integration (cf. [14, Cor. 4.1, p. 27]) along with (2.12) yields

$$|x(t)| \leq V_\gamma(t) \leq V_\gamma(\tau) - k(1-\gamma)\int_\tau^t |x(s)|\, ds \tag{2.15}$$

for all $t \geq \tau$. Therefore, $\int_0^\infty |x(t)|\, dt$ converges.

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$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{2.2}$$

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Part 3. [The zero solution of (2.2) is globally asymptotically stable and every solution is uniformly continuous.]

Recall $x(t) = x(t, t_0, \varphi)$ denotes the solution with $x(t) = \varphi(t)$ for $0 \le t \le t_0$. Consider again $V(t) = V(t, x(\cdot))$:

$$V(t) = |x(t)| + \int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du \right] |x(s)|\,ds.$$

By (2.7), namely $D_r V(t) \le 0$, $V(t)$ is decreasing on $[t_0, \infty)$. Consequently, as $V(t) \ge 0$,

$$\lim_{t\to\infty} V(t) \text{ exists and is finite.}$$

$\therefore V$ is uniformly continuous on $[0, \infty)$ by Lemma 2.1.

Now go back to $V_\gamma(t)$:

$$V_\gamma(t) := |x(t)| + \int_0^t \left[\gamma a(s) - \frac{1}{\gamma} \int_s^t |b(u,s)|\,du \right] |x(s)|\,ds.$$

An integration of

$$D_r V_\gamma(t) \le -(1-\gamma)a(t)|x(t)| \qquad \text{(cf. (2.13))}$$

yields

$$V_\gamma(t) - V_\gamma(\tau) \le -(1-\gamma) \int_\tau^t a(s)|x(s)|\,ds.$$

Hence,

$$\int_\tau^t a(s)|x(s)|\,ds \le \frac{V_\gamma(\tau)}{1-\gamma}$$

for all $t \ge \tau$. And so $a(t)|x(t)|$ is Riemann integrable on $[\tau,\infty)$. By Lemma 2.2,

$$\int_0^t a(s)|x(s)|\,ds$$

is uniformly continuous on $[0,\infty)$.

This suggests that the integral terms in $V(t)$, viz. the function

$$W(t) := \int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right] |x(s)|\,ds,$$

may also be uniformly continuous on $[0,\infty)$. Let's prove that this is the case. Define $h\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ by

$$h(t,s) := \begin{cases} a(s) - \int_s^t |b(u,s)|\,du & \text{if } t \ge s, \\ a(s) & \text{if } t < s. \end{cases} \tag{2.16}$$

For a fixed $t^* \in [0,\infty)$,

$$0 \le h(t^*,s)|x(s)| \le a(s)|x(s)|.$$

Thus, $h(t^*,s)|x(s)|$ is integrable on $[0,\infty)$ and

$$\int_0^\infty h(t^*,s)|x(s)|\,ds \le \int_0^\infty a(s)|x(s)|\,ds < \infty.$$

Consequently,

$$\int_0^\infty h(t,s)|x(s)|\,ds$$

defines a function, call it $w(t)$, on the interval $[0,\infty)$.

For $t_2 \geq t_1 \geq 0$, $h(t_2, s) \leq h(t_1, s)$. This implies the function

$$w(t) = \int_0^\infty h(t,s)|x(s)|\,ds$$

is decreasing on $[0,\infty)$. As $w(t) \geq 0$, $w(t)$ approaches a finite limit, say L, as $t \to \infty$. This in turn implies $W(t) \to L$ as

$$\begin{aligned}
|W(t) - L| &\leq |W(t) - w(t)| + |w(t) - L| \\
&= \left| \int_0^t h(t,s)|x(s)|\,ds - \int_0^\infty h(t,s)|x(s)|\,ds \right| \\
&\quad + \big|w(t) - L\big| \\
&= \left| - \int_t^\infty h(t,s)|x(s)|\,ds \right| + \big|w(t) - L\big| \\
&= \int_t^\infty a(s)|x(s)|\,ds + \big|w(t) - L\big|.
\end{aligned}$$

$\therefore W$ is uniformly continuous on $[0,\infty)$ by Lemma 2.1.

We established that V and W are uniformly continuous on $[0,\infty)$. Hence, so is the difference $V(t) - W(t) = |x(t)|$.

By Part 2, $|x| \in L^1[0,\infty)$.

By Barbălat's lemma, $|x(t)| \to 0$ as $t \to \infty$.

By Lemma 2.1, $x(t)$ is uniformly continuous on $[0,\infty)$. $\square$

Example 2.4. Every solution of

$$x'(t) = -(t+1)x(t) + \int_0^t \frac{2t}{(1+t^2-s^2)^2}\, x(s)\, ds \tag{2.17}$$

belongs to $L^1[0,\infty)$ and is uniformly continuous on $[0,\infty)$ and its zero solution is globally asymptotically stable.

Proof. As $a(t) = t+1$, (2.3) is satisfied with $k = 1$. And as $b(t,s) = 2t(1+t^2-s^2)^{-2}$,

$$\int_s^t |b(u,s)|\, du = \int_s^t \frac{2u}{(1+u^2-s^2)^2}\, du = 1 - \frac{1}{1+t^2-s^2}.$$

Clearly then (2.1) is satisfied as

$$\int_s^t |b(u,s)|\, du \le 1 + s = a(s)$$

for all $t \ge s \ge 0$. Also, for $t \ge s \ge 2$,

$$\int_s^t |b(u,s)|\, du \le 1 - \frac{1}{1+t^2-s^2} < 1 + \frac{1}{s} = \frac{1}{s}a(s) \le \frac{1}{2}a(s).$$

In other words, (2.4) is also satisfied with $t_1 = 2$ and $\lambda = 1/2$. ☐

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$$\int_s^t |b(u,s)|\, du \le a(s) \quad \forall\, t \ge s \ge 0. \tag{2.1}$$

For a $t_1 \ge 0$, $\exists\, k > 0$ and $\lambda \in (0,1)$ such that

$$a(t) \ge k, \quad \forall\, t \ge t_1 \tag{2.3}$$

$$\int_s^t |b(u,s)|\, du \le \lambda a(s), \quad \forall\, t \ge s \ge t_1. \tag{2.4}$$

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The Maple worksheet [3]—at the Maple Application Center website—uses the implicit trapezoidal rule and Newton's method for nonlinear systems to numerically approximate solutions of scalar Volterra integro-differential equations and draw their graphs. It was used to compute and graph the three numerical solutions of (2.17) in Fig. 2. A step size of $h = 0.1$ was used.

One solution satisfies the initial condition $x(0) = 1$.

The other two solutions have the initial functions:

$$\varphi(t) = 2 + t \text{ on the initial interval } [0, 1]$$

and

$$\varphi(t) = -2 + \sin(2t) \text{ on } [0, 2].$$

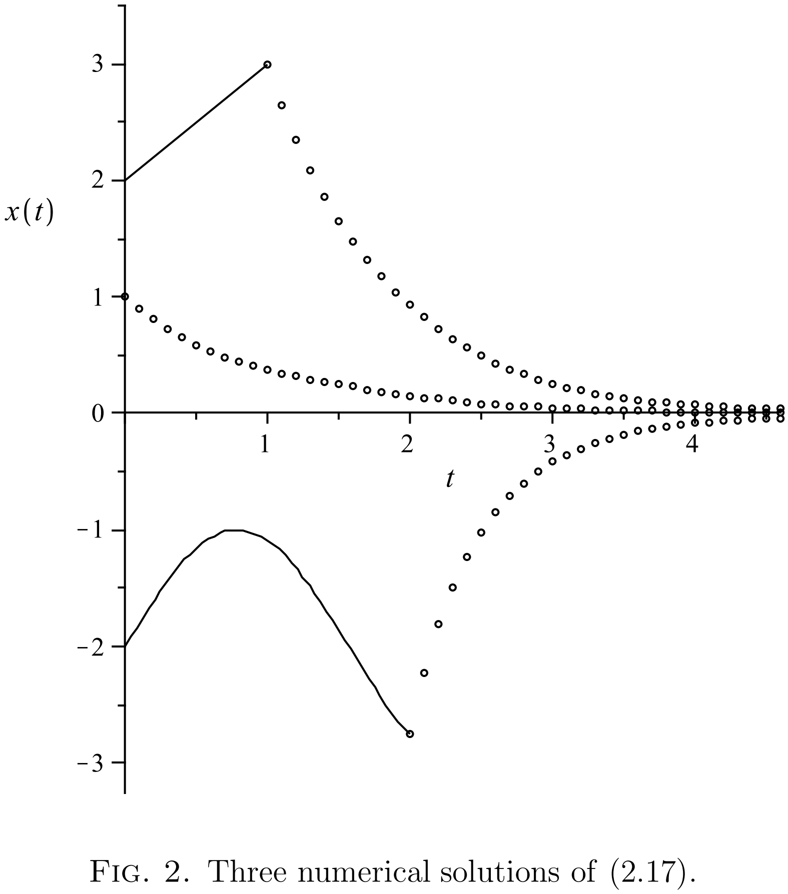

Fig. 2. Three numerical solutions of (2.17).

The next theorem drops the condition in Theorem 2.3 that $a(t)$ be eventually bounded below by a positive constant (cf. (2.3)).

Theorem 2.5. *Let* $a\colon [0,\infty) \to [0,\infty)$ *and* $b\colon \Omega \to \mathbb{R}$ *be continuous functions satisfying conditions* (2.1) *and* (2.4)*. If a constant* L *and a nonnegative function* $p \in L^1[0,\infty)$ *exist such that*

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\,d\xi \le L \tag{2.18}$$

for all $t \ge 0$ *and*

$$\int_s^t e^{-\int_\xi^t a(u)\,du}|b(\xi,s)|\,d\xi \le p(s) \tag{2.19}$$

for all $t \ge s \ge 0$*, then the zero solution of*

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{2.2}$$

is globally asymptotically stable.

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$$\int_s^t |b(u,s)|\,du \le a(s) \quad \forall\, t \ge s \ge 0. \tag{2.1}$$

For a $t_1 \ge 0$, $\exists\, \lambda \in (0,1)$ such that

$$\int_s^t |b(u,s)|\,du \le \lambda a(s), \quad \forall\, t \ge s \ge t_1. \tag{2.4}$$

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Example 2.6. The zero solution of

$$x' = -a(t)x \tag{2.20}$$

is globally asymptotically stable if a constant $\alpha > 0$ exists such that

$$\int_{t_0}^{t} a(u)\,du \geq \alpha(t - t_0) \tag{2.21}$$

for $t \geq t_0 \geq 0$.

Proof. Condition (2.18) is satisfied with $L = 1/\alpha$. The other three conditions in Theorem 2.5 are trivially satisfied as $b(t, s)$ in (2.2) is identically equal to zero. □

Remark 1. In fact under condition (2.21), the zero solution of (2.20) is uniformly asymptotically stable in the large (cf. [11, p. 88]).

Example 2.7. The zero solution of

$$x'(t) = -tx(t) + \int_0^t b(t,s)\,x(s)\,ds, \tag{2.22}$$

where $b\colon \Omega \to \mathbb{R}$ is the function

$$b(t,s) = \begin{cases} 0, & \text{if } 0 \le s < \frac{1}{3} \\ \dfrac{(3t-1)(3s-1)}{(1+t+s)^5}, & \text{if } s \ge \frac{1}{3}, \end{cases} \tag{2.23}$$

is globally asymptotically stable.

Proof. Since $a(t) = t$,

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\,d\xi = e^{-t^2/2}\int_0^t e^{\xi^2/2}\,d\xi = \sqrt{2}D(t/\sqrt{2}), \tag{2.24}$$

where D is Dawson's integral, namely,

$$D(t) := e^{-t^2}\int_0^t e^{\xi^2}\,d\xi. \tag{2.25}$$

It can be shown using an elementary argument that D is bounded on $[0,\infty)$, but that is a long established fact (cf. [1, p. 298]). From the information in [1] or through the use of a computer algebra system, we find that (2.24) has a absolute maximum value of $0.7651\ldots$ at $t = 1.3069\ldots$. Consequently, (2.18) holds with $L = 0.8$.

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There exists a constant L such that

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\,d\xi \le L, \quad \forall t \ge 0. \tag{2.18}$$

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For $t \geq s \geq 1/3$,

$$\begin{aligned}\int_s^t |b(u,s)|\,du &= \int_s^t \frac{(3u-1)(3s-1)}{(1+u+s)^5}\,du \\ &\leq (3s-1)\int_s^t \frac{(3u-1)}{(1+u)^5}\,du \leq \left[\frac{3s-1}{(1+s)^4}\right] s \leq 0.2s. \end{aligned} \tag{2.26}$$

Since $b(t,s) = 0$ for $0 \leq s < 1/3$, we conclude

$$\int_s^t |b(u,s)|\,du \leq \lambda a(s) \tag{2.27}$$

for all $t \geq s \geq 0$, where $\lambda = 0.2$. Thus, (2.1) and (2.4) hold for all $t \geq s \geq 0$.

From (2.26) we see that

$$\int_s^t e^{-\int_\xi^t a(u)\,du}|b(\xi,s)|\,d\xi \leq \int_s^t |b(\xi,s)|\,d\xi \leq p(s), \tag{2.28}$$

where

$$p(s) := \begin{cases} 0, & \text{if } 0 \leq s < \frac{1}{3} \\ \left[\dfrac{3s-1}{(1+s)^4}\right] s, & \text{if } s \geq \frac{1}{3}. \end{cases} \tag{2.29}$$

Since $p \in L^1[0,\infty)$, condition (2.19) holds. ☐

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There exists a nonnegative $p \in L^1[0,\infty)$ such that

$$\int_s^t e^{-\int_\xi^t a(u)\,du}|b(\xi,s)|\,d\xi \leq p(s), \quad \forall t \geq s \geq 0. \tag{2.19}$$

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The graphs of four numerical solutions of (2.22) computed with the Maple worksheet [3] are shown in Fig. 3. One of the solutions satisfies the initial condition $x(0) = 2$. The others issue from the initial functions

$$\varphi(t) = -5,$$

$$\varphi(t) = -2 + \sin(4t),$$

and

$$\varphi(t) = 3 + 2t$$

on the initial interval $[0, 1]$.

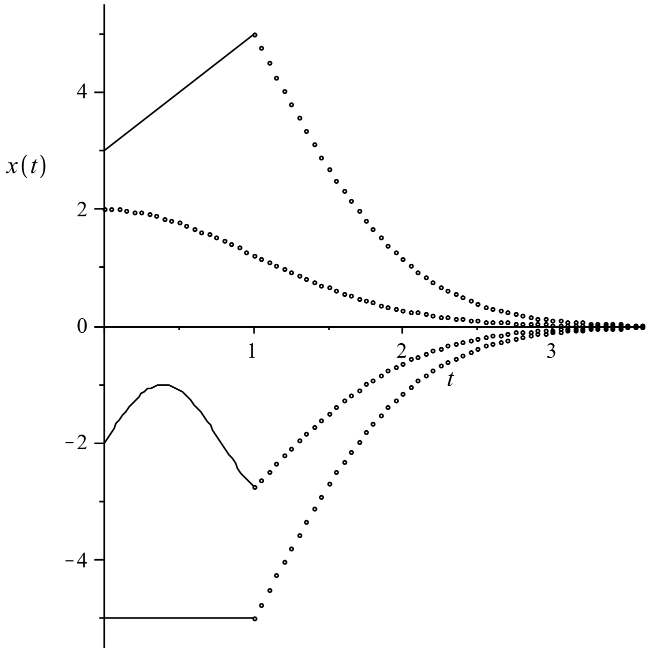

FIG. 3. Four numerical solutions of (2.22).

Proof of Theorem 2.5.

Proof. The zero solution of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{2.2}$$

is stable because of (2.1). We will show that all of its solutions approach zero as $t \to \infty$ by comparing them to the solutions of the equations

$$y'(t) = -\Big(a(t) + \frac{1}{k}\Big)y(t) + \int_0^t b(t,s)y(s)\,ds, \tag{2.30}$$

where $k \in \mathbb{N}$ (the set of natural numbers). Note that for a given $k \in \mathbb{N}$, (2.30) has a globally asymptotically stable zero solution on account of $a(t) + 1/k$ and $b(t,s)$ satisfying all of the conditions of Theorem 2.3.

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$$\int_s^t |b(u,s)|\,du \le a(s) \quad \forall\, t \ge s \ge 0. \tag{2.1}$$

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For any $t_0 \geq 0$ and $\varphi \in C[0, t_0]$, let $x(t)$ be the solution of

$$x'(t) = -a(t)x(t) + \int_0^t b(t, s)x(s)\, ds \tag{2.2}$$

with $x(t) = \varphi(t)$ for $0 \leq t \leq t_0$. For $k \in \mathbb{N}$, let $y_k(t)$ denote the solution of

$$y'(t) = -\Big(a(t) + \frac{1}{k}\Big)y(t) + \int_0^t b(t, s)y(s)\, ds \tag{2.30}$$

with the same initial function—i.e., $y_k(t) = \varphi(t)$ for $0 \leq t \leq t_0$.

Now consider the difference $x(t) - y_k(t)$. For $t \geq t_0$,

$$\begin{aligned} \frac{d}{dt}[x(t) - y_k(t)] = -a(t)[x(t) - y_k(t)] &+ \frac{1}{k}y_k(t) \\ &+ \int_0^t b(t, s)[x(s) - y_k(s)]\, ds. \end{aligned}$$

Multiplying this by

$$\mu(t) := \exp\left(\int_0^t a(v)\, dv\right)$$

and replacing $a(t)\mu(t)$ with $\mu'(t)$, we obtain

$$\frac{d}{dt}\left(\mu(t)[x(t) - y_k(t)]\right) = \frac{1}{k}\mu(t)y_k(t) + \mu(t)\int_0^t b(t, s)[x(s) - y_k(s)]\, ds.$$

Then an integration from t_0 to t yields

$$x(t) - y_k(t) = \frac{1}{k}\int_{t_0}^{t} \frac{\mu(\xi)}{\mu(t)} y_k(\xi)\, d\xi + \int_{t_0}^{t} \frac{\mu(\xi)}{\mu(t)} \int_0^{\xi} b(\xi, s)[x(s) - y_k(s)]\, ds\, d\xi$$

for all $t \geq t_0$. As $x(t) \equiv y_k(t)$ on $[0, t_0]$, it follows that

$$|x(t) - y_k(t)| \leq \frac{1}{k}\int_0^{t} \frac{\mu(\xi)}{\mu(t)} |y_k(\xi)|\, d\xi + \int_0^{t} \frac{\mu(\xi)}{\mu(t)} \int_0^{\xi} |b(\xi, s)|\, |x(s) - y_k(s)|\, ds\, d\xi \tag{2.31}$$

for all $t \geq 0$.

In the context of (2.30) $\left[y'(t) = -\Big(a(t) + \frac{1}{k}\Big)y(t) + \int_0^t b(t,s)y(s)\, ds\right]$, the inequality (2.10) is

$$|y_k(t)| \leq M_k(t_0)|\varphi|_{t_0}, \quad \text{for all } t \geq t_0, \tag{2.32}$$

where

$$M_k(t_0) := 1 + \int_0^{t_0} \left[a(s) + \frac{1}{k} - \int_s^{t_0} |b(u, s)|\, du\right] ds.$$

In fact (2.32) holds for all $t \geq 0$ as $y_k(t) \equiv \varphi(t)$ on $[0, t_0]$ and $M_k(t_0) \geq 1$.

Now observe that the upper bound in (2.32) can be replaced by one that is independent of k:

$$|y_k(t)| \leq M_1(t_0)|\varphi|_{t_0}$$

for all $k \in \mathbb{N}$ and $t \geq 0$.

Then $|y_k(t)| \leq M_1(t_0)|\varphi|_{t_0}$ and (2.18) imply

$$\int_0^t \frac{\mu(\xi)}{\mu(t)}\,|y_k(\xi)|\,d\xi = \int_0^t e^{-\int_\xi^t a(u)\,du}\,|y_k(\xi)|\,d\xi \leq LM_1(t_0)|\varphi|_{t_0} \tag{2.33}$$

for all $t \geq 0$. As for the iterated integral in (2.31), interchanging the order of integration and applying (2.19), we obtain

$$\begin{aligned} &\int_0^t \frac{\mu(\xi)}{\mu(t)} \int_0^\xi |b(\xi, s)|\,|x(s) - y_k(s)|\,ds\,d\xi \qquad (2.34)\\ &\qquad = \int_0^t \left(\int_s^t e^{-\int_\xi^t a(v)\,dv} |b(\xi, s)|\,d\xi \right) |x(s) - y_k(s)|\,ds \\ &\qquad \leq \int_0^t p(s)|x(s) - y_k(s)|\,ds. \end{aligned}$$

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There exists a constant L such that

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\,d\xi \leq L, \quad \forall t \geq 0. \tag{2.18}$$

There exists a nonnegative $p \in L^1[0, \infty)$ such that

$$\int_s^t e^{-\int_\xi^t a(u)\,du} |b(\xi, s)|\,d\xi \leq p(s), \quad \forall t \geq s \geq 0. \tag{2.19}$$

****

It follows then from (2.31), (2.33), and (2.34) that

$$|x(t) - y_k(t)| \leq \frac{1}{k} L M_1(t_0) |\varphi|_{t_0} + \int_0^t p(s)\, |x(s) - y_k(s)|\, ds$$

for $t \geq 0$. By Gronwall's inequality,

$$|x(t) - y_k(t)| \leq \frac{1}{k} L M_1(t_0) |\varphi|_{t_0}\, e^{\int_0^t p(s)\, ds}.$$

Consequently,

$$|x(t)| \leq |y_k(t)| + \frac{1}{k} L M_1(t_0) |\varphi|_{t_0}\, e^{\int_0^\infty p(s)\, ds} < \infty \tag{2.35}$$

for all $t \geq 0$. This with $y_k(t) \to 0$ as $t \to \infty$ for every $k \in \mathbb{N}$ implies that $x(t) \to 0$ as $t \to \infty$. $\square$

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