Jensen's Inequality and Liapunov's Direct Method

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1. Introduction

In the theory of Liapunov's direct method for finite delay equations, four distinct types of problems arise which have been the focus of much investigation over the last 50 years. We note here that, with the aid of Jensen's inequality, all four can be collapsed into basically the same kind of problem. All four types can arise from a simple scalar equation of the form

$$x'(t) = -a(t)x + b(t)x(t-1) - c(t)x^3$$

and their treatment leads the investigator easily to attack far more general problems in much the same way. The classical Liapunov functional for this equation usually appears in the form of

$$V(t, x_t) = x^2(t) + \int_{t-1}^{t} |b(s+1)|x^2(s)ds$$

or

$$V(t, x_t) = |x(t)| + \int_{t-1}^{t} |b(s+1)||x(s)|ds$$

with the required relations varying slightly depending on the choice of these two functionals; both have been used extensively and they have counterparts for systems in which a and b are matrices, while $c(t)x^3$ is replaced by a more general nonlinear vector-valued function.

For this equation it is usually assumed that $a(t)$ is positive and dominates $b(t)$ in some way, while the real focus is on $c(t)$. Interesting things happen when:

1. $b(t)$ is bounded and $c(t) = 1$; this is the classical uniform asymptotic stability theorem.

2. $c(t) = 1$, while b is not bounded.

3. $c(t) = 1/t$ and b is not bounded.

4. $c(t) = |\sin t| - \sin t$.

Our discussion here centers around certain functional differential equations but the basic difficulties we wish to attack can be seen in equations without delay. These will be used in our first two sections for simplicity. Let $f : R \times R^n \to R^n$ be continuous and consider the system of differential equations

$$x' = f(t, x) \tag{1.1}$$

in which $x' = dx/dt$. It is supposed that

$$f(t, 0) = 0 \tag{1.2}$$

so that $x(t) = 0$ is a solution. The basic problem we consider is finding conditions so that solutions starting near the zero solution converge to zero. In the theory of Liapunov's direct method, we seek a differentiable scalar function $V : R \times R^n \to [0, \infty)$, together with scalar functions $W_i : [0, \infty) \to [0, \infty)$, called wedges, which are continuous, $W_i(0) = 0$, and W_i strictly increasing. The idea is to choose functions so shrewdly that

$$W_1(|x|) \le V(t, x), V(t, 0) = 0 \tag{1.3}$$

and the derivative of V along a solution of (1.1) satisfies

$$V'_{(1.1)}(t, x(t)) = gradV \cdot f + \partial V/\partial t \le -W_2(|x|). \tag{1.4}$$

This can be obtained by the chain rule which involves only (1.1) and the partial derivatives of V. Thus, the virtue of the method is that (1.4) can be computed directly from the differential equation itself, rather than from the unknown solution.

An existence theorem is invoked to prove that a solution $x(t, t_0, x_0) =: x(t)$ exists through an arbitrary point (t_0, x_0). Then we integrate (1.4) and say that

$$V(t, x(t)) \le V(t_0, x_0) - \int_{t_0}^{t} W_2(|x(s)|)ds.$$

If we can show that the solution exists for all future time and if we assume that the solution is bounded strictly away from *zero*, then the integral of $W_2(|x(s)|)$ tends to $-\infty$, driving $V(t, x(t))$ to $-\infty$, a contradiction to (1.3) since $0 \le W_1(|x|) \le V(t, x)$.

If functions can be found satisfying (1.3) and (1.4), then V is called a Liapunov function for (1.1). Construction of Liapunov functions is an art, rather than a science, yet investigators have been very successful at constructing effective Liapunov functions. Also, there are three instances of note when Liapunov functions can be constructed:

(i) The total energy of the system can constitute a suitable Liapunov function. Indeed, the method is often referred to as energy methods.

(ii) For a linear constant coefficient system, the Liapunov function is found from the solution of algebraic equations.

(iii) A first integral of an equation can often serve as a Liapunov function.

Liapunov's original work appeared in 1892 in Russian. It was translated into French in 1907. A modern English translation was published in 1992 [10]; it includes a photograph of Liapunov and a short biography. We were introduced to Liapunov's direct method primarily through the book by Yoshizawa [16]. Our own point of view is conveyed through the two monographs [3] and [4]; the second of which is dedicated to Prof. Yoshizawa and contains a photograph of him.

2. The Annulus Argument

The purpose of this section is to introduce the reader to the hypothesis of Marachkoff [13], introduced in 1940, and which has been used in both ordinary and functional differential equations to prove asymptotic stability. It simply asks that the right-hand side of the differential equation be bounded when the state variable (x or x_t) is bounded. While it is useful, it is highly objectionable for two reasons:

(a) It drastically limits the class of problems which we can consider.

(b) It actually detracts from the very properties which often promote strong asymptotic stability. For example, solutions of the scalar equation $x' = -x$ go to zero exponentially, but those of $x' = -(1+t^2)x$ tend to zero far more quickly.

In our first result here we show the reader exactly how the hypothesis is used. The remainder of the paper is devoted to showing ways in which we can avoid that objectionable condition by the use of Jensen's inequality.

DEFINITION. The zero solution of (1.1) is *stable* if for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta > 0$ such that $[|x_0| < \delta, t \geq t_0]$ implies that $|x(t, t_0, x_0)| < \epsilon$.

DEFINITION. The zero solution of (1.1) is *asymptotically stable* if it is stable and if for each $t_0 \geq 0$ there is a $\mu > 0$ such that $|x_0| < \mu$ implies that $x(t, t_0, x_0) \to 0$ as $t \to \infty$.

Let us focus on (1.1) through (1.4). In the first part of our proof of Theorem 2.1 we will show that (1.3) and $V'(t, x(t)) \leq 0$ imply that the zero solution of (1.1) is stable. The early investigators always believed that (1.3) and (1.4) would suffice for asymptotic stability, but they do not, as the reader may see in Burton [4; p.230]. If we add a wedge above V then we can get a much stronger result, namely uniform asymptotic stability (which we do not define here). But, to this very day, investigators do not know exactly what is needed to conclude asymptotic stability. In 1940 Marachkoff offered the following result. Hatvani [7] presents many contexts in which Marachkoff's assumption is used.

THEOREM 2.1. Suppose there is a function V satisfying (1.3) and (1.4). If, in addition, f is bounded for x bounded, then the zero solution of (1.1) is asymptotically stable.

Proof. Let $\epsilon > 0$ and $t_0 \geq 0$ be given. From (1.3) we see that $V(t_0, 0) = 0$ and, by continuity of V, there is a $\delta > 0$ such that $|x_0| < \delta$ implies that $V(t_0, x_0) < W_1(\epsilon)$. By (1.4) we have

$$W_1(|x(t, t_0, x_0)|) \leq V(t, x(t, t_0, x_0)) \leq V(t_0, x_0) < W_1(\epsilon)$$

for $t \geq t_0$ so that $|x(t, t_0, x_0)| < \epsilon$ since W_1 is increasing and $V' \leq 0$. This proves that the zero solution is stable.

For the given $\epsilon > 0$ and $t_0 \geq 0$, let $\delta > 0$ be found for stability. We claim that if $|x_0| < \delta$, then $x(t) := x(t, t_0, x_0) \to 0$ as $t \to \infty$. By way of contradiction, if this is false then there is a $\gamma > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $|x(t_n)| \geq \gamma$. But from (1.3) and (1.4) we see that for $V(t) := V(t, x(t))$ we have

$$0 \leq V(t) \leq V(t_0) - \int_{t_0}^{t} W_2(|x(s)|)ds.$$

Clearly, if $|x(t)|$ remained larger than $\gamma/2$ on some interval $[t^*, \infty)$, then we would have

$$0 \leq V(t) \leq V(t^*) - \int_{t^*}^{t} W_2(\gamma/2)ds \to -\infty,$$

a contradiction. Thus, there is a sequence $\{s_n\} \uparrow \infty$ with $|x(s_n)| \leq \gamma/2$.

The solution is racing back and forth across an annulus with radii γ and $\gamma/2$. It is from this that the name *annulus argument* comes.

Here is the critical part in which Marachkoff's condition is used. Since $|x(t)| < \epsilon$, there is a number $M > 0$ with $|f(t, x(t))| \leq M$ for all $t \geq 0$. We will now show that there is a number $T > 0$ such that $|x(t)| > \gamma/2$ for $t_n \leq t \leq t_n + T$. For suppose there is a $T_n > t_n$ with $|x(T_n)| = \gamma/2$. We would then have

$$x(T_n) = x(t_n) + \int_{t_n}^{T_n} f(s, x(s))ds$$

so that

$$\gamma/2 \leq |x(T_n) - x(t_n)| \leq \int_{t_n}^{T_n} Mds = M(T_n - t_n)$$

or

$$T_n - t_n \geq \gamma/2M.$$

The desired number T can be taken as

$$T = \gamma/4M.$$

Now, renumber the sequence $\{t_n\}$ so that $t_n + T < t_{n+1}$. Then $V(t_n + T) - V(t_n) \leq -TW_2(\gamma/2)$ from (1.4). Hence, $V(t) \to -\infty$, a contradiction. The proof is complete.

There is also a functional differential equation analogue of Theorem 2.1: the zero solution of the delay equation (3.1) in Section 3 is asymptotically stable if a Marachkoff-like condition that F be bounded for x_t bounded holds and if a Liapunov functional $V(t, x_t)$ satisfying (1.3) and (1.4) exists. A precise statement and proof of this is found in Burton [4;p. 261, Theorem 4.2.5 (c)]. A proof that looks precisely like that of Theorem 2.1, apart from some notational adjustments, can also be given.

The entire point of this paper is to derive an alternative to the annulus argument which will not require that f be bounded for x bounded. The integral form of Jensen's inequality, stated in Section 3, plays a central role. We have mentioned before that Hatvani [7] presents many contexts in which the annulus argument is used in conjunction with Liapunov's direct method. A good research project might begin with a study of that paper to see how the results can be improved using Jensen's inequality.

3. Uniform Asymptotic Stability through Convex-Downward Wedges

The use of wedges is an inherent part of Liapunov's direct method. A wedge W becomes even more potent in a stability argument if it also happens to be convex downward; for then the problems mentioned in Section 2 can be avoided by means of Jensen's inequality. Fortunately, as we will see below, ordinary wedges can even be converted to convex-downward wedges. We prefer the term *convex downward* to the more widely used *convex* to describe a function whose graph has the property that every point on each of its chords lies either above or on the portion of the graph subtended by the chord; more precisely:

DEFINITION. A function $G : [a, b] \to (-\infty, \infty)$ is said to be *convex downward* on $[a, b]$ if

$$G((1-\lambda)t_1 + \lambda t_2) \leq (1-\lambda)G(t_1) + \lambda G(t_2)$$

whenever $t_1, t_2 \in [a, b]$ and $0 \leq \lambda \leq 1$.

Some authors, such as Natanson [14;p. 38], just use the midpoint of each chord in their definition: $G([t_1 + t_2]/2) \leq [G(t_1) + G(t_2)]/2$. Nevertheless, it implies the first inequality when G is continuous on $[a, b]$ (cf. Stromberg [15; p. 204]). When it comes to wedges, properties of convex-downward functions that are of particular use are:

(i) If G is differentiable on an open interval (a, b), then G is convex downward on (a, b) if and only if G' is nondecreasing on (a, b).

(ii) If $f : [a, b] \to (-\infty, \infty)$ is increasing, then $F(t) = \int_a^t f(u)du$ is convex downward on $[a, b]$.

NOTE. For a wedge W and a constant $H > 0$, it follows from (ii) that

$$W_1(r) := \frac{1}{H}\int_0^r W(s)ds$$

is a convex-downward wedge by virtue of W increasing. Moreover, by a mean value theorem for integrals, $W_1(r) \leq W(r)$ for $r \in [0, H]$. Consequently, a local result, such as

$$V'(t, x_t) \leq -W(|x|) \text{ for } |x(t)| \leq H,$$

may be replaced by $V'(t, x_t) \leq -W_1(|x(t)|)$. In other words, we may just as well assume that W is convex downward in the first place.

Applying Jensen's inequality to integrals with integrands containing convex-downward wedges will prove to be an invaluable tool. With such wedges in mind, we state Jensen's inequality as given by Natanson [14; pp. 45-6], adapting it slightly to suit our uses of it.

JENSEN'S INEQUALITY. Let $\Phi : [0, \infty) \to [0, \infty)$ be continuous and convex downward. If f, p are nonnegative functions on $[a, b]$, f measurable and finite almost everywhere, p, $f \cdot p$ are integrable, and $\int_a^b p(t)dt > 0$, then

$$\Phi\left[\int_a^b f(t)p(t)dt \Big/ \int_a^b p(t)dt\right] \leq \int_a^b \Phi(f(t))p(t)dt \Big/ \int_a^b p(t)dt.$$

In conjunction with Liapunov's direct method, we show how asymptotic stability results for certain functional differential equations can be obtained by choosing a wedge Φ, or W, that is convex downward. In many of our uses of Jensen's inequality, f is the composite of a suitably chosen continuous, nonnegative function and a solution $x(t)$ of a scalar functional differential equation. Typically, $f(t) = |x(t)|$ or $f(t) = x^2(t)$. For example, for $f(t) = |x(t)|$, the form in which Jensen's inequality will appear is

$$-\int_a^b p(t)W(|x(t)|)dt \leq -\int_a^b p(t)dt \cdot W\left(\int_a^b p(t)|x(t)|dt \Big/ \int_a^b p(t)dt\right).$$

Our actual study begins with a functional differential equation with finite delay

(3.1) $$x'(t) = F(t, x_t), F(t, 0) = 0, ' = d/dt$$

where $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$ with r a positive constant. We assume that we have found a Liapunov functional $V(t, x_t)$ and wedges W_i satisfying

(3.2) $$W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)| + \int_{t-r}^t W_3(|x(s)|)ds)$$

and

(3.3) $$V'(t, x_t) \leq -W_4(W_3(|x(t)|),$$

where W_4 is convex downward. Much can and should be said about the derivative of V. Its upper right-hand derivative can exist when V is only continuous. But for our purposes here we will assume that V is at least Lipschitz and usually differentiable. The reader is referred to Yoshizawa [16; p.186 ff.] for a discussion.

The form in which the right-hand side of (3.3) is written does not pose a problem since

$$V'(t, x_t) \leq -W_5(|x(t)|),$$

where W_5 is an arbitrary wedge, can always be replaced by an inequality of the form (3.3). Simply rewrite W_5 as

$$W_5(s) = W_5(W_3^{-1}(W_3(s))) = (W_5 \circ W_3^{-1})(W_3(s))$$

and then integrate the wedge $(W_5 \circ W_3^{-1})(s)$ as is done in the note following (ii). The result is a convex-downward wedge W_4 with $W_4(s) \leq (W_5 \circ W_3^{-1})(s)$.

For a constant $A > 0$, the set C_A denotes the open $A-$ball in the Banach space C of continuous functions $\phi : [-r, 0] \to R^n$ with the supremum norm $\|\phi\| = \sup_{-r \leq s \leq 0} |\phi(s)|$, where $|\cdot|$ is any convenient norm on R^n. As is common throughout the literature, $|\cdot|$ will also represent absolute value. We allow $A = \infty$ in which case C_A denotes the entire space C. It is assumed that $F : R \times C_A \to R^n$ is continuous and takes closed bounded sets into bounded sets. From standard existence results (cf. Burton[4; p. 186 ff.]), for each $(t_0, \phi) \in R \times C_A$ there is at least one solution $x(t) := x(t, t_0, \phi)$ defined on an interval $[t_0, \alpha)$ with $x_{t_0} = \phi$; if there is a $B < A$ with $|x(t)| < B$ so long as it is defined, then $\alpha = +\infty$.

DEFINITION. The zero solution of (3.1) is:

a) *stable* if for each $\epsilon > 0$ and for each $t_0 \in R$ there is a $\delta > 0$ such that

$$[\phi \in C_\delta, t \geq t_0] \implies |x(t, t_0, \phi)| < \epsilon.$$

b) *uniformly stable (US)* if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$[\phi \in C_\delta, t_0 \in R, t \geq t_0] \implies |x(t, t_0, \phi)| < \epsilon.$$

c) *asymptotically stable (AS)* if it is stable and if for each $t_0 \in R$ there is an $\eta > 0$ such that

$$[\phi \in C_\eta] \implies |x(t, t_0, \phi)| \to 0 \text{ as } t \to \infty.$$

If, in addition, every solution $x(t, t_0, \phi)$, where $\phi \in C$, tends to zero, then the zero solution is *asymptotically stable in the large* or *globally asymptotically stable.*

d) *uniformly asymptotically stable (UAS)* if it is uniformly stable and if there is a $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that

$$[\phi \in C_\gamma, t_0 \in R, t \geq t_0 + T] \implies |x(t, t_0, \phi)| < \mu.$$

LEMMA 1. Suppose there is a differentiable functional V and wedges W_i so that on $R \times C_A$ we have

(i) $W_5(|x(t)|) \leq V(t, x_t) \leq W_6(\|x_t\|)$

and

(ii) $V'_{(3.1)}(t, x_t) \leq 0.$

Then the zero solution of (3.1) is uniformly stable.

Proof. Let $\epsilon < A$ be given and find $\delta > 0$ so that $W_6(\delta) < W_5(\epsilon)$. Then for

$$\|\phi\| < \delta, t_0 \in R, t \geq t_0$$

we have from (ii) that

$$W_5(|x(t, t_0, \phi)|) \leq V(t, x_t) \leq V(t_0, \phi) \leq W_6(\|\phi\|) < W_6(\delta)$$

so $|x(t, t_0, \phi)| < \epsilon$. This completes the proof.

EXERCISE. Show that if (3.2) holds then (i) holds.

REMARK. It was long conjectured that

(I) $W_1(|x(t)|) \leq V(t, x_t) \leq W_2(\|x_t\|)$

and

(II) $V'_{(3.1)}(t, x_t) \leq -W_3(|x(t)|)$ implied UAS; but it does not. Exactly 100 years after the publication of Liapunov's original article, Kato [8] produced a counterexample. A considerably simplified version was given by Makay [11].

Strengthening (I) to (3.2) does suffice for UAS and that is the basic result on the subject. It may be found in Burton [2] with a different proof. Much earlier Krasovskii [9] had obtained the weaker conclusion of asymptotic stability under the same conditions.

The proof plays the upper wedge against the derivative; this idea was introduced in the original proof [2] and has been used very effectively since then in other problems.

REMARK. In any uniform asymptotic stability proof, there are many details which need to be correctly put in order before the actual argument begins. In the proof below, the critical argument begins with the use of intervals defined in (3.7). The argument we see there is the one which will be repeated, with additional conditions, throughout the rest of the paper.

THEOREM 3.1. If (3.2) and (3.3) hold on $R \times C_A$, then the zero solution of (3.1) is uniformly asymptotically stable.

Proof. For $A/2 > 0$ find B of uniform stability; that is, for a solution $x(t, t_0, \phi)$ with an initial function ϕ satisfying $\|\phi\| < B$, then $|x(t, t_0, \phi)| < A/2$ for $t \geq t_0$. Denote by $x(t) := x(t, t_0, \phi)$ any fixed but arbitrary such solution.

Let $\epsilon > 0$ be given. We will find $T > 0$ such that $[\|\phi\| < B, t_0 \in R, t \geq t_0 + T]$ imply that $|x(t)| < \epsilon$. Notice that by (3.2) we have

$$V(t_0, \phi) \leq W_2(B + rW_3(B)) =: L \tag{3.4}$$

and let

$$M := W_1(\epsilon). \tag{3.5}$$

We will find a T so that for any such solution $x(t)$ we have

$$V(t_0 + T, x_{t_0+T}) < M$$

and, hence,

$$W_1(|x(t)|) \leq V(t) < W_1(\epsilon), \tag{3.6}$$

for $t \geq t_0 + T$, where $V(t) := V(t, x_t)$.

Consider the set of intervals

$$[t_{n-1}, t_n] := [t_0 + 2(n-1)r, t_0 + 2nr] \tag{3.7}$$

for $n = 1, 2, 3, ...$ We will find a positive integer N so that $V(t_N) < W_1(\epsilon)$. Consider the intervals for which this does not hold. In that case,

$$W_1(\epsilon) \leq V(t) \leq W_2\bigg(|x(t)| + \int_{t-r}^{t} W_3(|x(s)|)ds\bigg);$$

equivalently,

$$|x(t)| + \int_{t-r}^{t} W_3(|x(s)|)ds \geq 2K \tag{3.8}$$

for some $K > 0$.

Integrate (3.3) and obtain

$$V(t_n) - V(t_{n-1}) \leq -\int_{t_{n-1}}^{t_n} W_4(W_3(|x(s)|))ds \leq -\int_{t_n-r}^{t_n} W_4(W_3(|x(s)|))ds.$$

There are two possibilities:

1. $|x(t)| \geq K$ on $[t_n - r, t_n]$, so $V(t_n) - V(t_{n-1}) \leq -rW_4(W_3(K)) =: -D$;

or

2. there is a $t_n^* \in [t_n - r, t_n]$ with $|x(t_n^*)| < K$ so that

$$\int_{t_n^*-r}^{t_n^*} W_3(|x(s)|)ds > K.$$

In the latter case we have by Jensen's inequality,

$$V(t_n^*) - V(t_n^* - r) \leq -rW_4(\int_{t_n^*-r}^{t_n^*} W_3(|x(s)|)ds/r) < -rW_4(K/r) =: -Q.$$

Notice that $t_{n-1} \leq t_n^* - r \leq t_n^* \leq t_n$ so that $V(t_n) - V(t_{n-1}) \leq V(t_n^*) - V(t_n^* - r)$ since $V' \leq 0$. If we let $C = \min[D, Q]$, then $V(t_n) - V(t_{n-1}) \leq -C$. Take $NC > L - M$ so that $V(t_N) \leq V(t_{N-1}) - C \leq ... \leq V(t_0) - NC \leq L - NC < M$. In other words, $V(t_0 + T) < W_1(\epsilon)$, where $T = 2Nr$.

This completes the proof.

EXAMPLE 3.1. Consider the scalar equation

$$x' = -a(t)x + b(t)x(t - r) - c(t)x^3 \tag{3.9}$$

where $a, b, c : R \to R$ are continuous functions. Suppose that $a(t) > 0$ and constants $\beta > 0$ and $m \geq 0$ exist such that

$$\int_t^{t+r} b^2(s)ds \leq \beta \text{ and } 2a(t) - |b(t)| - |b(t+r)| \geq m$$

for $t \geq 0$. If [$m > 0$ and $c(t) \geq 0$] or if [$m = 0$ and $c(t) \geq \lambda$, λ a positive constant], then the zero solution of (3.9) is uniformly asymptotically stable.

Proof. The result follows from applying Theorem 3.1 to the classical Liapunov functional

$$V(t, x_t) = x^2(t) + \int_{t-r}^{t} |b(s+r)|x^2(s)ds.$$

To see this, along a solution of (3.9), we have

$$V'(t, x_t) = -2a(t)x^2(t) + 2b(t)x(t)x(t-r) - 2c(t)x^4(t) + |b(t+r)|x^2(t) - |b(t)|x^2(t-r)$$

$$\leq -2a(t)x^2(t) + |b(t)|x^2(t) + |b(t)|x^2(t-r)| + |b(t+r)|x^2(t) - |b(t)|x^2(t-r) - 2c(t)x^4(t)$$

$$\leq -(2a(t) - |b(t)| - |b(t+r)|)x^2(t) - 2c(t)x^4(t)$$

$$\leq -mx^2(t) - 2c(t)x^4(t).$$

It is an interesting and important exercise to reconcile the wedges in this example with those in Theorem 3.1.

By Schwarz's inequality,

$$V(t, x_t) \leq x^2(t) + \left(\int_{t-r}^{t} b^2(s+r)ds\right)^{1/2} \left(\int_{t-r}^{t} x^4(s)ds\right)^{1/2}$$

$$\leq x^2(t) + \beta^{1/2}\left(\int_{t-r}^{t} x^4(s)ds\right)^{1/2}$$

$$\leq kx^2(t) + k\left(\int_{t-r}^{t} x^4(s)ds\right)^{1/2},$$

where $k := \max[1, \beta^{1/2}]$.

Define $W(u) = \max[u^{1/2}, u^2]$. Then,

$$V(t, x_t) \leq kW(|x(t)|) + kW\left(\int_{t-r}^{t} x^4(s)ds\right)$$

$$\le 2kW\left(|x(t)| + \int_{t-r}^{t} x^4(s)ds\right),$$

as $W(u) + W(v) \le 2W(u+v)$. It follows that

$$W_1(|x(t)|) \le V(t, x_t) \le W_2\left(|x(t)| + \int_{t-r}^{t} W_3(|x(s)|)ds\right),$$

where $W_1(u) = u^2$, $W_2(u) = 2kW(u)$, and $W_3(u) = u^4$, showing that (3.2) holds.

Finally, define the wedge W_5 by $W_5(u) = mu^2$ if $m > 0$ and $c(t) \ge 0$ or by $W_5(u) = 2\lambda u^4$ if $m = 0$ and $c(t) \ge \lambda$. Then as $V'(t, x_t) \le -W_5(|x(t)|)$, it follows from previous remarks that a convex-downward wedge W_4 can be found so that (3.3) is satisfied in a neighborhood of the zero solution of (3.9). This concludes the proof.

It is easy to see that this stability result holds for any equation

$$x' = -a(t)x(t) + b(t)x(t-r) - c(t)x^{2n+1}$$

where n is a positive integer.

4. Constructing Upper Bounds on Liapunov Functionals with Jensen's Inequality

We are now going to go through a series of examples, adding some degree of difficulty each time. Since so many of the classical problems can be seen in the equation

$$x' = -a(t)x + b(t)x(t-1) - c(t)x^3, \tag{3.9}$$

we focus on (3.9) in the rest of the paper, setting $r = 1$ for the sake of simplicity. The reader can easily adapt the computations for

$$x' = -a(t)x^3(t) + b(t)x^3(t-1) - c(t)x^{2n+1}$$

by using the Liapunov functional

$$V(t, x_t) = (1/4)x^4 + (1/2)\int_{t-1}^{t} |b(s+1)|x^6(s)ds.$$

It is very interesting to compare the results, especially for $n = 0$, in the latter equation.

The integral condition in Example 3.1 was crucial in proving the uniform asymptotic stability of the zero solution of (3.9) in that it led to an upper bound on the Liapunov functional. With the aid of Jensen's inequality, other integral conditions may be found that also lead to upper bounds from which asymptotic stability results can be obtained. Our goal in the next example is to demonstrate this by inserting a suitably chosen function in a polynomial bounding the derivative V' of a Liapunov functional V and then integrating V' and using Jensen's inequality to obtain an upper bound on V itself.

EXAMPLE 4.1. Consider the scalar equation

$$x' = -a(t)x + b(t)x(t-1) - x^3 \tag{4.1}$$

where $a, b : R \to R$ are continuous, b vanishes at no more than a countably infinite number of points in every bounded interval, and

$$2a(t) \geq |b(t)| + |b(t+1)|. \tag{4.2}$$

If the function

$$B(t) := \left(\int_{t-1}^{t} b^2(s+1)ds \right)^{-1}$$

satisfies the condition

$$\sum_{i=1}^{\infty} B(t_i) = \infty \tag{4.3}$$

for every nondecreasing sequence $\{t_i\}_{i=1}^{\infty} \uparrow \infty$ satisfying $t_{i+1} - t_i \leq 2$, then every solution of (4.1) tends to zero as $t \to \infty$.

Proof. Using the Liapunov functional

$$V(t, x_t) = x^2 + \int_{t-1}^{t} |b(s+1)|x^2(s)ds, \tag{4.4}$$

the derivative of V along a solution of (4.1) is

$$V'(t, x_t) \leq -2a(t)x^2(t) + |b(t)|x^2(t) + |b(t)|x^2(t-1) + |b(t+1)|x^2(t) - |b(t)|x^2(t-1) - 2x^4(t)$$

or

$$V'(t, x_t) \leq -2x^4(t) \leq -(x^2(t))^2. \tag{4.5}$$

Integration of (4.5) will yield $x^4(t)$ integrable. However, this alone does not guarantee that $x(t)$ will go to zero since Marachkoff's condition that the right-hand side of (4.1) be bounded for x_t bounded is no longer one of the hypotheses. In fact, the divergent series condition (4.3) allows $b(t)$ to grow (e.g., see (4.13)) while there is no upper bound restriction at all on $a(t)$.

At this point, obtaining asymptotic stability by integrating V' seems futile–and yet, one of the strategies in Liapunov's direct method is to use the upper bound on V found by integrating V', along with V itself, to show that a zero solution exhibits some kind of stability. Basically the root of the problem is that the integrand in the Liapunov functional (4.4) contains the function b whereas the integral of (4.5) does not. In other words, we need an upper bound on V that includes

$$\int_{t-1}^{t} |b(s+1)|x^2(s)ds.$$

Nevertheless, there is a way of obtaining such an upper bound by essentially inserting b into the right-hand side of (4.5) before integrating.

First, define the function

$$\hat{b}(t) := \begin{cases} b(t), & \text{if } b(t) \neq 0 \\ 1, & \text{if } b(t) = 0 \end{cases}$$

and rewrite (4.5) as

$$V'(t, x_t) \leq -\hat{b}^2(t+1)\left(\frac{x^2}{|\hat{b}(t+1)|}\right)^2. \tag{4.6}$$

Now integrate both sides from $t-1$ to t and then use Jensen's inequality. In the statement of Jensen's inequality given in Section 3, set $p(t) = \hat{b}^2(t+1)$ and $f(t) = x^2(t)/|\hat{b}(t+1)|$. Since f, p, and $f \cdot p$ are continuous a.e. on $[t-1, t]$, they are measurable functions. Moreover, as the latter two are also bounded on the interval, they are also Riemann integrable. Letting $\Phi(u) = u^2$, we apply Jensen's inequality to the integral of the first term of the right-hand side of (4.6). Note that the resulting integrals do not change their values if $b(s+1)$ is substituted for $\hat{b}(s+1)$, since they are equal except for at most countably many $s \in [t-1, t]$. With this substitution and letting $V(t) := V(t, x_t)$, we obtain

$$V(t) - V(t-1) \leq -B(t)\left(\int_{t-1}^{t} |b(s+1)|x^2(s)ds\right)^2. \tag{4.7}$$

With the aid of (4.7), we can now show that an arbitrary solution tends to zero as $t \to \infty$. Imagine that we have a solution $x(t)$ on an interval, say $[0, \infty)$. Since the Liapunov functional along $x(t)$ is given by (4.4), we have

$$x^2(t) \le V(t)$$

and so it suffices to show that $V(t) \to 0$ as $t \to \infty$. By way of contradiction, if $V(t)$ does not tend to zero, then as $V' \le 0$, there is a positive constant c with

$$V(t) = x^2 + \int_{t-1}^{t} |b(s+1)| x^2(s) ds \ge c. \tag{4.8}$$

Consider the decreases in V over the successive intervals $[i-1, i]$ for $i = 2, 3, 4, ...$ If there is not a point $t_i \in [i-1, i]$ with $x(t_i) \le c/2$, then by (4.5) we have

$$V(i) - V(i-1) < -c^2/4; \tag{4.9}$$

but if there is such a point, then from (4.8) we have

$$\int_{t_i-1}^{t_i} |b(s+1)| x^2(s) ds \ge c/2. \tag{4.10}$$

In the latter case, it follows that $V(t_i) - V(t_i - 1) \le -B(t_i)c^2/4$, or as $i-2 \le t_i - 1 < t_i \le i$,

$$V(i) - V(i-2) \le -B(t_i)c^2/4. \tag{4.11}$$

We first note that (4.9) can hold for at most a finite number of values of i. To be definite, for a fixed solution, there is an integer N so that (4.9) fails for all $i \ge N$ and so (4.11) holds from that point on. The terms of the sequence $\{t_i\}_{i=N}^{\infty}$ satisfy $t_{i+1} - t_i \le 2$ and approach ∞ since $t_i \in [i-1, i]$. As a result, by (4.3) the sum from $i = N$ to ∞ of the right-hand side of (4.11) is $-\infty$. This forces $V(t) \to -\infty$, a contradiction. This concludes the proof.

Consider the sentence after (3.9) with the new and more general equation. You can get a hint about solving it by examining (4.6) above.

We would like to find a simple sufficient condition for (4.3) to hold. One that comes to mind is for $B(t) \geq 1/[k(t+1)]$ or

$$\int_{t-1}^{t} b^2(s+1)ds \leq k(t+1) \tag{4.12}$$

for $t \geq M$, where $M \geq 0$ and $k > 0$ are constants. Then, for a nondecreasing sequence $\{t_i\}_{i=0}^{\infty} \uparrow \infty$ with $t_0 \in [n, n+1]$, n a nonnegative integer, and $t_{i+1} - t_i \leq 2$, we have

$$B(t_i) \geq \frac{1}{k(t_i+1)} \geq \frac{1}{k(n+2(i+1))}$$

for $t_i \geq M$. By the integral test,

$$\sum_{i=0}^{\infty} \frac{1}{k(n+2(i+1))} = \infty,$$

from which condition (4.3) follows. It is also worth noting that the condition

$$|b(t+1)| \leq c(t+1)^{1/2}, \tag{4.13}$$

for $t \geq M$ and a constant $c > 0$, implies (4.12) and hence (4.3), since $t+1/2 = \int_{t-1}^{t}(s+1)ds$.

In view of existing literature, is this a good condition? It is, indeed. Burton and Makay [5] use very sophisticated methods to find conditions for asymptotic stability. One of them is that $a(t) + |b(t)| \leq c(t+1)\ln(t+2)$ for some constant $c > 0$. Consequently, if $b(t) = c\ln(t+2)$, then this condition requires that $a(t) \leq ct\ln(t+2)$. However, note that $b(t) = c\ln(t+2)$ easily satisfies condition (4.13) and all that is required for all solutions to approach zero as $t \to \infty$ is that (4.2) hold, namely, $2a(t) \geq |b(t)| + |b(t+1)|$.

Whenever we write specific functions in an equation instead of working with general functions, results are almost always greatly improved. To understand what is happening in such problems we propose the following exercises:

1. Rework Example 4.1, obtaining a counterpart for (4.13) when (4.1) is replaced by

$$x' = -a(t)x + b(t)x(t-1) - x^7$$

2. Assume that you have some general equation and have constructed a Liapunov functional yielding the pair

$$W_1(|x|) \leq V(t, x_t) \leq W_2(|x|) + W_3\left(\int_{t-1}^{t} |b(s)|W_4(|x(s)|)ds\right)$$

and

$$V'(t, x_t) \le -W_5(|x|).$$

Follow the work in Example 4.1 to obtain an asymptotic stability result based on this pair.

5. Asymptotic Stability through Divergent Series

The asymptotic stability result for (4.1) invites us to look for similar results for

$$x' = -a(t)x + b(t)x(t-1) - c(t)x^{2n+1}$$

when $n > 1$ and the function c is different from $c(t) \equiv 1$ and not constant. Such a function will now appear explicitly in the upper bound on the derivative of the Liapunov functional (4.4). This then brings up questions regarding the use of Jensen's inequality, how this changes the divergent series condition (4.3), and whether other conditions are needed. Rather than answering these questions directly, we invite the reader to investigate them at this point. The information gleaned from doing this leads to even more general results, such as the next theorem.

THEOREM 5.1. Let

$$x'(t) = F(t, x_t),\ \ F(t, 0) = 0 \tag{5.1}$$

be a functional differential equation with finite delay $r > 0$, where $F : R \times C_A \to R^n$ is continuous and takes closed bounded sets into bounded sets. Suppose a differentiable functional V and wedges W_i exist that satisfy

$$W_1(|x(t)|) \le V(t, x_t) \le W_2(|x(t)|) + W_3\left(\int_{t-r}^{t} |\lambda(s)| W_4(|x(s)|) ds\right) \tag{5.2}$$

and

$$V'_{(5.1)}(t, x_t) \le -\eta(t)\big(W_4(|x(t)|)\big)^2 \tag{5.3}$$

on $R \times C_A$, where $\lambda, \eta : R \to R$ are continuous functions with η nonnegative. In addition, suppose that a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_{i+1} - t_i \ge 2r$ and a constant $\alpha > 0$ exist so that

$$\int_{t_i - r}^{t_i} \eta(t) dt \ge \alpha \tag{5.4}$$

and that on each of the intervals $[t_i - 2r, t_i]$, η is nowhere zero whereas λ is nonzero a.e. If, for every sequence $\{\tau_i\}_{i=1}^{\infty}$ with $\tau_i \in [t_i - r, t_i]$, the series with terms

$$B(\tau_i) := \left(\int_{\tau_i - r}^{\tau_i} \frac{\lambda^2(t)}{\eta(t)} dt \right)^{-1} \tag{5.5}$$

diverges to ∞, i.e.,

$$\sum_{i=1}^{\infty} B(\tau_i) = \infty, \tag{5.6}$$

then the zero solution of (5.1) is asymptotically stable. Moreover, if $C_A = C$, the zero solution is globally asymptotically stable.

Proof. For a given $t_0 \in R$, define the wedge W by

$$W(u) := W_2(u) + W_3\left(W_4(u) \int_{t_0 - r}^{t_0} |\lambda(s)| ds \right). \tag{5.7}$$

For $\epsilon \in (0, A)$, where $0 < A \leq \infty$, find a $\delta > 0$ so that $W(\delta) < W_1(\epsilon)$. For an initial function $\phi \in C_\delta$, let $x(t) := x(t, t_0, \phi)$ be a solution of (5.1). For $t \geq t_0$, it follows from (5.2) and (5.3) that

$$W_1(|x(t)|) \leq V(t, x_t) \leq V(t_0, \phi) < W(\delta) < W_1(\epsilon)$$

and so $|x(t)| < \epsilon$ whenever $t \geq t_0$. Thus, $x(t)$ is defined on $[t_0, \infty)$ and the zero solution is stable.

Next, we prove that $x(t)$ tends to the zero solution as $t \to \infty$. Since $V(t) := V(t, x_t) \geq W_1(|x(t)|)$, we can prove this by arguing that $V(t) \to 0$. If this were not the case, then as $V'(t) \leq 0$,

$$V(t) \geq \mu \text{ for } t \geq t_0 \tag{5.8}$$

for some constant $\mu > 0$. Select a term t_n from $\{t_i\}_{i=1}^{\infty}$ so that $t_n \geq t_0 + 2r$. On each interval $[t_i - r, t_i]$ with $i \geq n$, either

1. $W_2(|x(t)|) > \mu/2$, or
2. $W_2(|x(\tau_i)|) \leq \mu/2$ for some $\tau_i \in [t_i - r, t_i]$.

In the first case, for all t in that interval we have $W_4(|x(t)|) > W_4(W_2^{-1}(\mu/2)) =: k$. Consequently, from (5.3) and (5.4), it follows that

$$V(t_i) - V(t_i - r) < -k^2 \int_{t_i-r}^{t_i} \eta(t)dt \le -\alpha k^2. \tag{5.9}$$

As for the second case, (5.2) implies that

$$\int_{\tau_i-r}^{\tau_i} |\lambda(s)|W_4(|x(s)|)ds \ge W_3^{-1}(\mu/2) =: K. \tag{5.10}$$

Define the function $\hat{\lambda}$ by $\hat{\lambda}(t) = 1$ if $\lambda(t) = 0$; otherwise, $\hat{\lambda}(t) = \lambda(t)$. For $t \in [t_i - 2r, t_i]$, we can rewrite (5.3) as

$$V'(t) \le -\frac{\hat{\lambda}^2(t)}{\eta(t)} \left(\frac{\eta(t)}{|\hat{\lambda}(t)|} W_4(|x(t)|) \right)^2$$

since $\eta(t) \ne 0$. Integrating from $\tau_i - r$ to τ_i, we obtain

$$V(\tau_i) - V(\tau_i - r) \le -\int_{\tau_i-r}^{\tau_i} \frac{\hat{\lambda}^2(t)}{\eta(t)} \left(\frac{\eta(t)}{|\hat{\lambda}(t)|} W_4(|x(t)|) \right)^2 dt. \tag{5.11}$$

Now apply Jensen's inequality with $\Phi(t) = t^2$, $p(t) = \hat{\lambda}^2(t)/\eta(t)$, and $f(t) = \frac{\eta(t)}{|\hat{\lambda}(t)|} W_4(|x(t)|)$. Since $\hat{\lambda} = \lambda$ a.e., we replace $\hat{\lambda}$ with λ in the resulting integrals to obtain

$$V(\tau_i) - V(\tau_i - r) \le -B(\tau_i) \left(\int_{\tau_i-r}^{\tau_i} |\lambda(t)| W_4(|x(t)|)dt \right)^2.$$

Since $t_i - 2r \le \tau_i - r < \tau_i \le t_i$, from $V'(t) \le 0$ and (5.10) it follows that

$$V(t_i) - V(t_i - 2r) \le -B(\tau_i)K^2. \tag{5.12}$$

For $i \ge n$, either (5.9) or (5.12) holds at each t_i. However, as V is always decreasing, (5.9) can hold at only finitely many of these t_i since V is nonnegative. Consequently, (5.12) must hold at the remaining t_i. Summing both sides of (5.12) over these infinitely many t_i and using (5.6), we conclude that $V(t) \to -\infty$ as $t \to \infty$; but this contradicts (5.8). Hence, $V(t) \to 0$ as $t \to \infty$ and so $|x(t)| \to 0$. In fact, since this argument depends only on a solution $x(t, t_0, \phi)$ existing on the entire interval $[t_0, \infty)$, we have proved that every bounded solution approaches the zero solution as $t \to \infty$.

To sum up, the zero solution is asymptotically stable since for each $t_0 \in R$ and $\epsilon \in (0, A)$, there is a $\delta > 0$ such that $\phi \in C_\delta$ implies that $|x(t, t_0, \phi)| < \epsilon$ for $t \geq t_0$ and $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$. Finally, for the case $A = \infty$, every solution $x(t, t_0, \phi)$, for any $\phi \in C$, is bounded since $|x(t)| < W_1^{-1}(W(\|\phi\|)$ for all $t \geq t_0$. Consequently, every solution approaches the zero solution as $t \to \infty$. This concludes the proof.

COROLLARY 5.1. Suppose that a differentiable functional V and wedges W_i exist for (5.1) satisfying (5.2) and (5.3) on $R \times C_A$, where λ, $\eta : R \to R$ are continuous functions and $\eta(t) \geq |\lambda(t)|$. Furthermore, suppose that a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_{i+1} - t_i \geq 2r$ and a constant $\alpha > 0$ exist so that (5.4) holds and

$$\int_{\tau-r}^{\tau} \eta(t)dt > 0 \tag{5.13}$$

for all $\tau \in [t_i - r, t_i]$. If, for every sequence $\{\tau_i\}_{i=1}^{\infty}$ with $\tau_i \in [t_i - r, t_i]$, the series with terms

$$B(\tau_i) := \left(\int_{\tau_i - r}^{\tau_i} \eta(t)dt \right)^{-1} \tag{5.14}$$

diverges to ∞, then the zero solution of (5.1) is AS and globally asymptotically stable if $C_A = C$.

Proof. Since $|\lambda(t)| \leq \eta(t)$, it follows from (5.2) that

$$W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)|) + W_3\left(\int_{t-r}^{t} \eta(s) W_4(|x(s)|)ds \right). \tag{5.15}$$

Since the same function, namely η, now appears in both right-hand sides of (5.3) and (5.15), we can integrate the upper bound on V' directly without first having to insert a function as is done in the proof of Theorem 5.1. With condition (5.13), all of the criteria of Jensen's inequality are met. The result of integrating (5.3) and using Jensen's inequality is

$$V(\tau_i) - V(\tau_i - r) \leq -B(\tau_i) \left(\int_{\tau_i - r}^{\tau_i} \eta(t) W_4(|x(t)|)dt \right)^2,$$

where (5.14) replaces the $B(\tau_i)$ of (5.5). Aside from these relatively minor changes, the rest of the proof is exactly the same as that of Theorem 5.1.

COROLLARY 5.2. Suppose a differentiable functional V and wedges W_i exist for (5.1) that satisfy (5.2) and

$$V'_{(5.1)}(t, x_t) \leq -\eta(t)W_4(|x(t)|) \tag{5.16}$$

on $R \times C_A$, where $\lambda, \eta : R \to R$ are continuous functions and $\eta(t) \geq 0$. Also, suppose that a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_{i+1} - t_i \geq 2r$ exists, along with a constant $\alpha > 0$, so that

$$\int_{t_i - r}^{t_i} \eta(t)dt \geq \alpha \tag{5.17}$$

and that there is a constant $L > 0$, such that for every sequence $\{\tau_i\}_{i=1}^{\infty}$ with $\tau_i \in [t_i - r, t_i]$,

$$L|\lambda(t)| \leq \eta(t) \tag{5.18}$$

for $t \in [\tau_i - r, \tau_i]$. Then the zero solution of (5.1) is AS, or globally asymptotically stable if $C_A = C$.

Proof. The proof is identical to that of Theorem 5.1 up through (5.10), except that (5.9) must be slightly altered to

$$V(t_i) - V(t_i - r) < -\alpha k \tag{5.19}$$

on account of the different upper bound on V'. The integral of (5.16) along a solution from $\tau_i - r$ to τ_i together with (5.10) and (5.18) yields

$$V(\tau_i) - V(\tau_i - r) \leq -\left(\int_{\tau_i - r}^{\tau_i} \eta(t)W_4(|x(t)|)dt\right) \leq -L\left(\int_{\tau_i - r}^{\tau_i} |\lambda(t)|W_4(|x(t)|)dt\right) \leq -LK.$$

Accordingly, the inequality

$$V(t_i) - V(t_i - 2r) \leq -LK \tag{5.20}$$

replaces (5.12). Thus, either (5.19) or (5.20) holds at each t_i. Aside from not having to deal with a divergent series, the rest of the proof is the same as before.

EXERCISE. Consider again Theorem 3.1. Can we obtain UAS when $V' \leq -c(t)W_4(W_3(|x(t)|))$ for $c(t) \geq 0$ and

$$\int_{t-r}^{t} c(s)ds \geq \alpha > 0?$$

6. Examples

EXAMPLE 6.1. Consider the scalar equation

$$x' = -a(t)x + b(t)x(t-r) - c(t)x^{2n+1}, \tag{6.1}$$

where n is a positive integer and $a, b, c : R \to R$ are continuous. Let c be nonnegative and

$$2a(t) \geq |b(t)| + |b(t+r)|. \tag{6.2}$$

Suppose that a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_{i+1} - t_i \geq 2r$ exists such that $\lambda(t) := b(t+r)$ and $\eta(t) := c(t)$ satisfy all of the conditions associated with (5.4) and (5.5) in Theorem 5.1. Furthermore, for $n > 1$ suppose that a constant $\beta > 0$ exists such that

$$\int_t^{t+r} |b(s)|ds \leq \beta \tag{6.3}$$

for $t \in R$. Then the zero solution of (6.1) is globally asymptotically stable.

Proof. The derivative of the Liapunov functional

$$V(t, x_t) = x^2(t) + \int_{t-r}^{t} |b(s+r)|x^2(s)ds, \tag{6.4}$$

along a solution of (6.1) is

$$V'(t, x_t) \leq -2c(t)x^{2n+2}(t) \leq -c(t)\big(|x(t)|^{n+1}\big)^2. \tag{6.5}$$

Thus, (5.3) holds with $W_4(u) = u^{n+1}$. We can use Jensen's inequality to find a wedge W_3 to show that (5.2) also holds. Take $p(t) = |b(t+1)|$ and $f(t) = |x(t)|^{n+1}$. Define $\Phi(t) = t^{\frac{2}{n+1}}$ so that $\Phi(f(t)) = x^2(t)$. Note, however, that as Φ' is decreasing when $n > 1$, Φ is concave downward; that is, $-\Phi$ is convex downward on $[0, \infty)$. Consequently, use of this particular Φ reverses the inequality with the result that

$$\int_{t-r}^{t} |b(s+r)|x^2(s)ds \leq \left(\int_{t-r}^{t} |b(s+r)|ds\right)^{\frac{n-1}{n+1}} \left(\int_{t-r}^{t} |b(s+r)|W_4(|x(s)|)ds\right)^{\frac{2}{n+1}}. \tag{6.6}$$

This inequality trivially holds for $n = 1$. This, together with (6.3), shows that (5.2) holds for $n \geq 1$ with $W_3(u) = \beta^{\frac{n-1}{n+1}} u^{\frac{2}{n+1}}$ and $W_1(u) = W_2(u) = u^2$. Thus, all of the conditions in Theorem 5.1 are satisfied.

Remark. When $n = 1$, it follows from (6.6) that the integral condition (6.3) is actually unnecessary; in this case, $W_3(u) = u$. Wedges of the type $W(u) = u^m$ are convex downward for $1 \le m < \infty$ but concave downward for $0 < m \le 1$. So, W_3 acts as a boundary between the two types.

EXAMPLE 6.2. The zero solution of

$$x' = -(t^2 + 3)x(t) + 2(t-1)(\sin t)x(t-1) - t^2 \sin^2(t+1)x^3(t) \tag{6.7}$$

is globally asymptotically stable.

Proof. Referring to Example 6.1, we have $|b(t)| + |b(t+1)| \le 2|t-1| + 2|t|$. Thus,

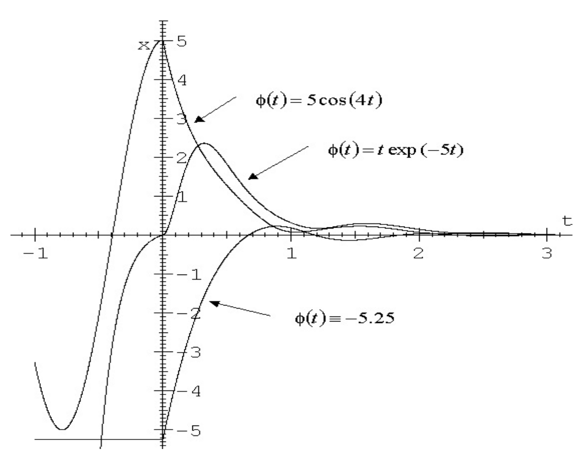
$$\eta(t) := 2 - |b(t)| - |b(t+1)| \ge 2(t^3 + 3) - 2(|t-1| + |t|) \ge 2.$$

Letting $\alpha = 2$, the integral inequality (5.4) holds for any sequence $\{t_i\}_{i=1}^{\infty}$. By defining $t_i = (2i-1)\frac{\pi}{2}$, the condition that $c(t) \ne 0$ on the intervals $[t_i - 2, t_i]$ is satisfied. Since $b^2(t+1) = 4c(t)$, it follows that

$$B(\tau_i) := \left(\int_{\tau_i - 1}^{\tau_i} \frac{b^2(t+1)}{c(t)} dt \right)^{-1} = \frac{1}{4}$$

for any $\tau_i \in [t_i - 1, t_i]$. Consequently, (5.6) holds. Since $n = 1$, condition (6.3) is unnecessary, which concludes the proof.

Solutions of (6.7) corresponding to three initial functions on $[-1, 0]$ were graphed below with the numerical solver *DifEqu*, written by Makay [12].



We now give an example of coefficients for (6.1) that are unbounded but vanish on a sequence of infinitely many intervals.

EXAMPLE 6.3. Let H be a square wave function defined by

$$H(t) = \sum_{i=1}^{\infty}(-1)^{i+1}u(t-0.5i),$$

where $u(t-T)$ is the unit step function defined by $u(t-T)=0$ if $t \le T$, $u(t-T)=1$ if $t > T$. The zero solution of

$$x' = -(|t|+|t+1|)|\sin 2\pi t|H(t)x(t) - 2t\sin(2\pi t)H(t-1)x(t-1) - 2|t+1|x^3(t) \tag{6.8}$$

is globally asymptotically stable.

Proof. Let a, b, and c denote the coefficients as in (6.1) where $n=1$. We show that the conditions in Corollary 5.1 are satisfied using the Liapunov functional (6.4) with $r=1$. Clearly, (5.2) holds with $\lambda(t) := b(t+1)$, $W_i(u) = u^2$ for $i = 1, 2, 4$, and $W_3(u) = u$. Since the period of both H and $\sin 2\pi t$ is 1,

$$\begin{aligned}|b(t)| + |b(t+1)| &= 2|t||\sin 2\pi t|H(t-1) + 2|t+1||\sin 2\pi t|H(t)\\ &\le 2(|t|+|t+1|)|\sin 2\pi t|H(t) = 2a(t).\end{aligned}$$

Hence, it follows (cf. (6.2) and (6.5)) that (5.3) holds with $\eta(t) := c(t) = 2|t+1|$. The condition that $\eta(t) \ge |\lambda(t)|$ is met since $|\lambda(t)| = |b(t+1)| \le 2|t+1|$. Now define the sequence $\{t_i\}_{i=1}^{\infty}$ by letting $t_i = 2i$. From

$$\int_{\tau-1}^{\tau} \eta(t)dt = \int_{\tau-1}^{\tau} 2(t+1)dt = 2\tau + 1$$

for $\tau \ge t_1 - 1 = 1$, it follows that (5.13) and (5.4) with $\alpha = 5$ hold. Finally, the divergent series condition is satisfied as

$$B(\tau_i) = \left(\int_{\tau_i-1}^{\tau_i} \eta(t)dt\right)^{-1} = \frac{1}{2\tau_i+1} \ge \frac{1}{4(i+1)}$$

and so $\sum_{i=1}^{\infty} B(\tau_i) = \infty$.

A prototypic example demonstrating Liapunov theory applied to delay differential equations is the scalar linear equation with constant coefficients

$$x' = -ax(t) + bx(t-r), \text{ where } a > |b| > 0. \tag{6.9}$$

The classical Liapunov functionals in this paper can be used to show that its zero solution is uniformly asymptotically stable (cf. Burton [4; p. 252] or Driver [6; p. 368]). As a matter of fact, this result follows from Example 3.1. Moreover, Driver (cf. p. 244) uses a classical Liapunov funcional to show that every solution of (6.9) is bounded and then uses this in an ad hoc argument to prove that every solution in fact tends to zero exponentially as $t \to \infty$. The next example, a consequence of Corollary 5.2, can also be used to prove that every solution tends to zero. Even though it does not show that the rate of decay is exponential, it does give asymptotic stability results for certain linear delay equations with variable coefficients.

EXAMPLE 6.4. Let $a, b : R \to R$ be continuous. If for $\eta(t) := 2a(t) - |b(t)| - |b(t+r)|$, positive constants α and L exist such that

$$\int_{t-r}^{t} \eta(s)ds \geq \alpha \tag{6.10}$$

and

$$L|b(t+r)| \leq \eta(t) \tag{6.11}$$

for $t \geq t_1$, for some $t_1 \in R$, then the zero solution of

$$x' = -a(t)x + b(t)x(t-r) \tag{6.12}$$

is globally asymptotically stable.

Proof. Using the Liapunov functional (6.4), condition (5.2) holds with $W_i(u) = u^2$ $(i = 1, 2, 4), W_3(u) = u$, and $\lambda(t) = |b(t+r)|$. Condition (5.16) holds since

$$V'_{(6.12)}(t, x_t) \leq -(a(t) - |b(t)| - |b(t+r)|)x^2(t) = -\eta(t)W_4(|x(t)|),$$

where $\eta(t) = 2a(t) - |b(t)| - |b(t+r)|$. Since (6.10) and (6.11) hold for all $t \geq t_1$, the sequential conditions in Corollary 5.2 are met, completing the proof.

EXAMPLE 6.5. The zero solution of (6.9) is globally asymptotically stable.

Proof. This follows directly from Example 6.4. Since $\eta(t) \equiv 2(a - |b|)$, (6.10) and (6.11) hold with $\alpha = 2(a - |b|)r$ and $L = (a - |b|)/|b|$, respectively.

EXAMPLE 6.6. The zero solution of

$$x' = -(\sin 2\pi t + k|\sin 2\pi t|)x(t) + (\sin 2\pi t + |\sin 2\pi t|)x(t-1) \tag{6.11}$$

is globally asymptotically stable if $k > 1$.

Proof. Referring again to Example 6.4, $\eta(t) = 2a(t) - |b(t)| - |b(t+r)| = 2(k-1)|\sin 2\pi t|$ and

$$\int_{t-1}^{t} \eta(s)ds = 2(k-1)\int_{t-1}^{t} |\sin 2\pi s|ds = 4(k-1)\int_{0}^{1/2} \sin 2\pi s ds = \frac{4}{\pi}(k-1).$$

Thus, for $k > 1$, (6.10) and (6.11) are satisfied with $\alpha = \frac{4}{\pi}(k-1)$ and $L = k-1$, respectively.

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