Fast Fourier Transform

Using FourierTransform with 2m points where m=2^p.

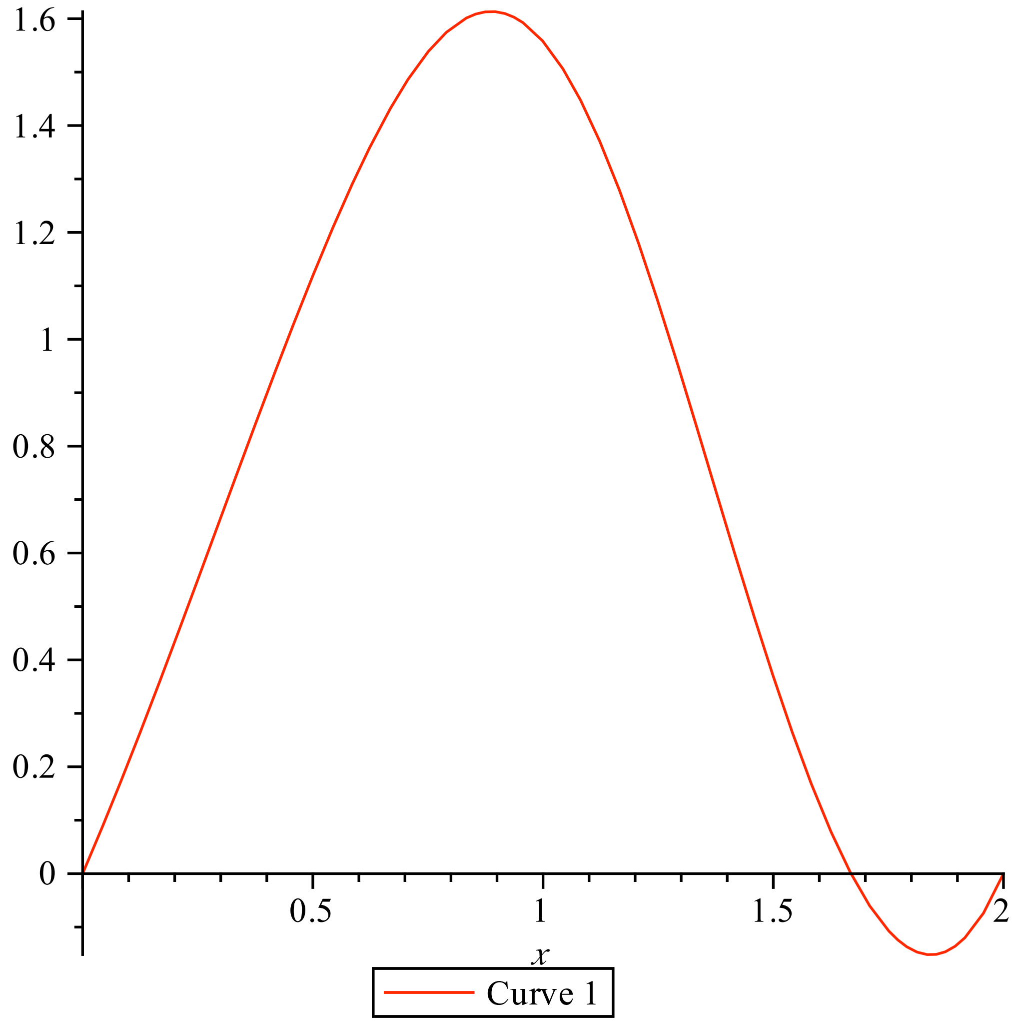
```
> restart;
```

We do Example 2 on page 378. We begin by entering our function and plotting it.

```
> f:=x->x^4-3*x^3+2*x^2-tan(x*(x-2));
```

$$f := x \to x^4 - 3\,x^3 + 2\,x^2 - \tan(x\,(x-2))$$

```
> plot(f(x),x=0..2);
```



We note that our original function f(x) is not periodic. However, f(x) restricted to [0,2] is a good building block (rubber stamp) for a periodic function since f(2)=f(0) and even, as we show below, f ' (2)=f ' (0). Thus it makes sense to do trigonometric interpolation.

```
> fp:=D(f);
```

$$fp := x \to 4\,x^3 - 9\,x^2 + 4\,x - \left(1 + \tan(x\,(x-2))^2\right)\,(-2 + 2\,x)$$

```
> evalb(fp(0)=fp(2));
```

true

We load two packages **LinearAlgebra** and DiscreteTransforms .

```
> with(LinearAlgebra):with(DiscreteTransforms):
```

The **fast Fourier transform** requires that we use a number of points that is a power of 2. We enter the value of p such that we have m= 2^p and where we have 2m = 2^{p+1} data points (x_j, y_j). The degree of our trigonometric polynomial will be $m = 4$.

```
> p:=2;m:=2^p;
```

$$p := 2$$
$$m := 4$$

We input the endpoints of our closed interval [A,B].

```
> A:=0;B:=2;
```

$$A := 0$$
$$B := 2$$

We form the array x of equally spaced x values.

```
> for j from 0 to 2*m-1 do
  x[j]:=A+j*(B-A)/(2*m)
  od;
```

$$x_0 := 0$$
$$x_1 := \frac{1}{4}$$
$$x_2 := \frac{1}{2}$$
$$x_3 := \frac{3}{4}$$
$$x_4 := 1$$
$$x_5 := \frac{5}{4}$$
$$x_6 := \frac{3}{2}$$
$$x_7 := \frac{7}{4}$$

We enter our y values into the y array either by using a function that generates them or by putting them into a list L and using a for loop to place them into the array.

```
> for j from 0 to 2*m-1 do
  y[j]:=evalf(f(x[j]))
  od;
```

$$y_0 := 0.$$
$$y_1 := 0.5497612755$$
$$y_2 := 1.119096460$$
$$y_3 := 1.537853226$$
$$y_4 := 1.557407725$$
$$y_5 := 1.069103226$$

$$y_6 := 0.3690964599$$

$$y_7 := -0.1064887245$$

We use a linear transformation z = T(x) = -Pi+(2Pi)*(x-A)/(B-A) to transform the interval [A, B] into the interval [-Pi, Pi]. We use the linear transformation to transform each x_j in [A, B] into the corresponding z_j in [-Pi, Pi].

```
> eq:=z=-Pi+(2*Pi)*(x-A)/(B-A);
> for j from 0 to 2*m-1 do
  z[j]:=-Pi+(2*Pi)*(x[j]-A)/(B-A)
  od;
```

$$eq := z = -\pi + \pi x$$

$$z_0 := -\pi$$

$$z_1 := -\frac{3}{4}\pi$$

$$z_2 := -\frac{1}{2}\pi$$

$$z_3 := -\frac{1}{4}\pi$$

$$z_4 := 0$$

$$z_5 := \frac{1}{4}\pi$$

$$z_6 := \frac{1}{2}\pi$$

$$z_7 := \frac{3}{4}\pi$$

We need to make a vector of our y values (real or complex).

```
> Y:=Vector(2*m,[seq(y[j],j=0..2*m-1)]);
```

$$Y := \begin{bmatrix} 0. \\ 0.5497612755 \\ 1.119096460 \\ 1.537853226 \\ 1.557407725 \\ 1.069103226 \\ 0.3690964599 \\ -0.1064887245 \end{bmatrix}$$

We now compute the transform using the Maple command FourierTransform .

```
> Z:=evalf(FourierTransform(Y));
```

$$Z := \begin{bmatrix} 2.155201240 + 0.\,I \\ -1.091547757 - 0.5464150430\,I \\ 0.02447112902 - 0.06629126074\,I \\ -0.009705806460 - 0.01608495702\,I \\ -0.001636371699 + 0.\,I \\ -0.009705806460 + 0.01608495702\,I \\ 0.02447112902 + 0.06629126074\,I \\ -1.091547757 + 0.5464150430\,I \end{bmatrix}$$

At this point that the array Z contains the c_k.

```
> for k from 0 to m do
  a[k]:=Re((-1)^k*sqrt(2./m)*Z[k+1]);
  b[k]:=Im(((-1)^(k+1)*sqrt(2./m)*Z[k+1]));
  od;
```

$$a_0 := 1.523957412$$
$$b_0 := -0.$$
$$a_1 := 0.7718408210$$
$$b_1 := -0.3863737823$$
$$a_2 := 0.01730370127$$
$$b_2 := 0.04687500000$$
$$a_3 := 0.006863041565$$
$$b_3 := -0.01137378218$$
$$a_4 := -0.001157089525$$
$$b_4 := -0.$$

We now form the trigonometric polynomial over the interval [-Pi, Pi] in the variable z. We name this polynomial s.

```
> s:=z->a[0]/2+a[m]/2*cos(m*z)+sum('a[k]*cos(k*z)+b[k]*sin(k*z)',
  'k'=1..m-1);
> s:=unapply(s(z),z);;
```

$$s := z \to \frac{1}{2}\,a_0 + \frac{1}{2}\,a_m \cos(m\,z) + \sum_{'k'=1}^{m-1} 'a_k \cos(k\,z) + b_k \sin(k\,z)'$$

$$s := z \to 0.7619787060 - 0.0005785447625\cos(4\,z) + 0.7718408210\cos(z) - 0.3863737823\sin(z) + 0.01730370127\cos(2\,z) + 0.04687500000\sin(2\,z) + 0.006863041565\cos(3\,z) - 0.01137378218\sin(3\,z)$$

We translate the trigonometric polynomial back to our original interval [A, B].

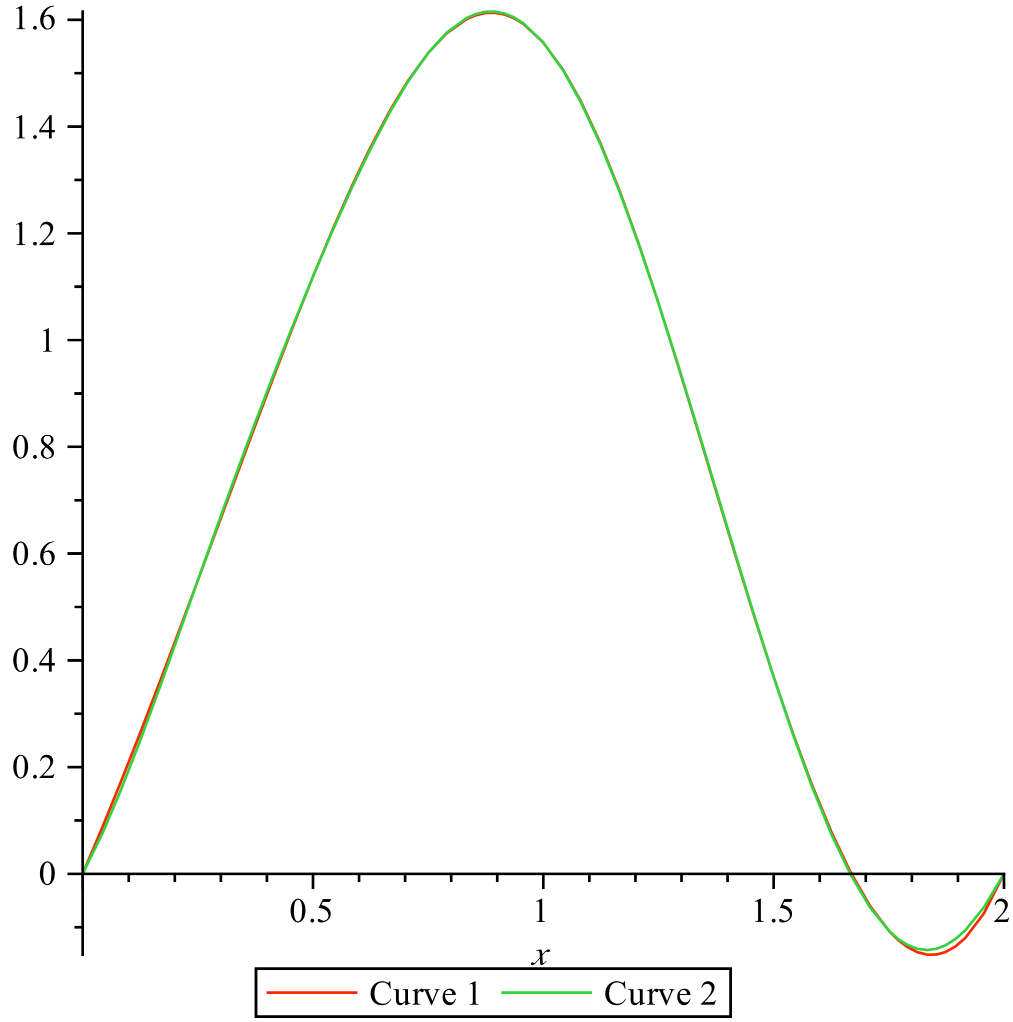
```
> Z:=rhs(eq);
> S:=unapply(s(Z),x);
```

$$Z := -\pi + \pi\,x$$

$$S := x \rightarrow 0.7619787060 - 0.0005785447625 \cos(4 \pi x) - 0.7718408210 \cos(\pi x) + 0.3863737823 \sin(\pi x) + 0.01730370127 \cos(2 \pi x) + 0.04687500000 \sin(2 \pi x) - 0.006863041565 \cos(3 \pi x) + 0.01137378218 \sin(3 \pi x)$$

We graph the function (red) and the trigonometric polynomial (green) on the interval [A, B].

```
> plot({f(x),S(x)},x=A..B);
```



Using FourierTransform with 2m points for any m > 0.

We use m = 20.

```
> m:=20;
```

$$m := 20$$

```
> for j from 0 to 2*m-1 do
x[j]:=A+j*(B-A)/(2*m)
od;
```

$$x_0 := 0$$

$$x_1 := \frac{1}{20}$$

$$x_2 := \frac{1}{10}$$

$$x_3 := \frac{3}{20}$$

$$x_4 := \frac{1}{5}$$

$$x_5 := \frac{1}{4}$$

$$x_6 := \frac{3}{10}$$

$$x_7 := \frac{7}{20}$$

$$x_8 := \frac{2}{5}$$

$$x_9 := \frac{9}{20}$$

$$x_{10} := \frac{1}{2}$$

$$x_{11} := \frac{11}{20}$$

$$x_{12} := \frac{3}{5}$$

$$x_{13} := \frac{13}{20}$$

$$x_{14} := \frac{7}{10}$$

$$x_{15} := \frac{3}{4}$$

$$x_{16} := \frac{4}{5}$$

$$x_{17} := \frac{17}{20}$$

$$x_{18} := \frac{9}{10}$$

$$x_{19} := \frac{19}{20}$$

$$x_{20} := 1$$

$$x_{21} := \frac{21}{20}$$

$$x_{22} := \frac{11}{10}$$

$$x_{23} := \frac{23}{20}$$

$$x_{24} := \frac{6}{5}$$

$$x_{25} := \frac{5}{4}$$

$$x_{26} := \frac{13}{10}$$

$$x_{27} := \frac{27}{20}$$

$$x_{28} := \frac{7}{5}$$

$$x_{29} := \frac{29}{20}$$

$$x_{30} := \frac{3}{2}$$

$$x_{31} := \frac{31}{20}$$

$$x_{32} := \frac{8}{5}$$

$$x_{33} := \frac{33}{20}$$

$$x_{34} := \frac{17}{10}$$

$$x_{35} := \frac{7}{4}$$

$$x_{36} := \frac{9}{5}$$

$$x_{37} := \frac{37}{20}$$

$$x_{38} := \frac{19}{10}$$

$$x_{39} := \frac{39}{20}$$

We enter our y values into the y array either by using a function that generates them or by putting them into a list L and using a for loop to place them into the array.

```
> for j from 0 to 2*m-1 do
  y[j]:=evalf(f(x[j]))
  od;
```

$$y_0 := 0.$$
$$y_1 := 0.1024413825$$
$$y_2 := 0.2094198376$$
$$y_3 := 0.3202307958$$
$$y_4 := 0.4340028516$$
$$y_5 := 0.5497612755$$
$$y_6 := 0.6664587156$$
$$y_7 := 0.7829824200$$
$$y_8 := 0.8981438222$$
$$y_9 := 1.010654978$$
$$y_{10} := 1.119096460$$
$$y_{11} := 1.221882631$$
$$y_{12} := 1.317232349$$
$$y_{13} := 1.403155707$$
$$y_{14} := 1.477469381$$
$$y_{15} := 1.537853226$$
$$y_{16} := 1.581957491$$
$$y_{17} := 1.607562030$$
$$y_{18} := 1.612776741$$
$$y_{19} := 1.596258374$$
$$y_{20} := 1.557407725$$
$$y_{21} := 1.496508374$$
$$y_{22} := 1.414776741$$
$$y_{23} := 1.314312030$$
$$y_{24} := 1.197957491$$
$$y_{25} := 1.069103226$$
$$y_{26} := 0.9314693810$$
$$y_{27} := 0.7889057070$$
$$y_{28} := 0.6452323490$$
$$y_{29} := 0.5041326310$$
$$y_{30} := 0.3690964599$$

$$y_{31} := 0.2434049775$$
$$y_{32} := 0.1301438222$$
$$y_{33} := 0.0322324200$$
$$y_{34} := -0.0475412844$$
$$y_{35} := -0.1064887245$$
$$y_{36} := -0.1419971484$$
$$y_{37} := -0.1515192042$$
$$y_{38} := -0.1325801624$$
$$y_{39} := -0.08280861754$$

We use a linear transformation z = T(x) = -Pi+(2Pi)*(x-A)/(B-A) to transform the interval [A, B] into the interval [-Pi, Pi]. We use the linear transformation to transform each x_j in [A, B] into the corresponding z_j in [-Pi, Pi].

```
> eq:=z=-Pi+(2*Pi)*(x-A)/(B-A);
> for j from 0 to 2*m-1 do
  z[j]:=-Pi+(2*Pi)*(x[j]-A)/(B-A)
  od;
```

$$eq := z = -\pi + \pi x$$
$$z_0 := -\pi$$
$$z_1 := -\frac{19}{20}\pi$$
$$z_2 := -\frac{9}{10}\pi$$
$$z_3 := -\frac{17}{20}\pi$$
$$z_4 := -\frac{4}{5}\pi$$
$$z_5 := -\frac{3}{4}\pi$$
$$z_6 := -\frac{7}{10}\pi$$
$$z_7 := -\frac{13}{20}\pi$$
$$z_8 := -\frac{3}{5}\pi$$
$$z_9 := -\frac{11}{20}\pi$$
$$z_{10} := -\frac{1}{2}\pi$$

$$z_{11} := -\frac{9}{20}\pi$$

$$z_{12} := -\frac{2}{5}\pi$$

$$z_{13} := -\frac{7}{20}\pi$$

$$z_{14} := -\frac{3}{10}\pi$$

$$z_{15} := -\frac{1}{4}\pi$$

$$z_{16} := -\frac{1}{5}\pi$$

$$z_{17} := -\frac{3}{20}\pi$$

$$z_{18} := -\frac{1}{10}\pi$$

$$z_{19} := -\frac{1}{20}\pi$$

$$z_{20} := 0$$

$$z_{21} := \frac{1}{20}\pi$$

$$z_{22} := \frac{1}{10}\pi$$

$$z_{23} := \frac{3}{20}\pi$$

$$z_{24} := \frac{1}{5}\pi$$

$$z_{25} := \frac{1}{4}\pi$$

$$z_{26} := \frac{3}{10}\pi$$

$$z_{27} := \frac{7}{20}\pi$$

$$z_{28} := \frac{2}{5}\pi$$

$$z_{29} := \frac{9}{20}\pi$$

$$z_{30} := \frac{1}{2}\pi$$

$$z_{31} := \frac{11}{20}\pi$$

$$z_{32} := \frac{3}{5}\pi$$

$$z_{33} := \frac{13}{20}\pi$$

$$z_{34} := \frac{7}{10}\pi$$

$$z_{35} := \frac{3}{4}\pi$$

$$z_{36} := \frac{4}{5}\pi$$

$$z_{37} := \frac{17}{20}\pi$$

$$z_{38} := \frac{9}{10}\pi$$

$$z_{39} := \frac{19}{20}\pi$$

```
> Y:=Vector(2*m,[seq(y[j],j=0..2*m-1)]);
```

$$Y := \left[\begin{array}{c} \mathit{1 .. 40\ Vector}_{column} \\ \mathit{Data\ Type: anything} \\ \mathit{Storage: rectangular} \\ \mathit{Order: Fortran\_order} \end{array}\right]$$

Maple decides not to print out the vector, but gives us some information on it. We now compute the transform using the command FourierTransform .

```
> Z:=evalf(FourierTransform(Y));
```

$$Z := \left[\begin{array}{c} \mathit{1 .. 40\ Vector}_{column} \\ \mathit{Data\ Type: anything} \\ \mathit{Storage: rectangular} \\ \mathit{Order: Fortran\_order} \end{array}\right]$$

At this point that the array Z contains the c_k.

```
> for k from 0 to m do
  a[k]:=Re((-1)^k*sqrt(2./m)*Z[k+1]);
  b[k]:=Im(((-1)^(k+1)*sqrt(2./m)*Z[k+1]));
  od;
```

$$a_0 := 1.524054433$$

$$b_0 := -0.$$

$$a_1 := 0.7717267078$$

$$b_1 := -0.3870174297$$
$$a_2 := 0.01747923756$$
$$b_2 := 0.04837532270$$
$$a_3 := 0.006478440403$$
$$b_3 := -0.01433101752$$
$$a_4 := -0.0005687667748$$
$$b_4 := 0.006043110066$$
$$a_5 := 0.0003637157810$$
$$b_5 := -0.003090990282$$
$$a_6 := -0.0001502368434$$
$$b_6 := 0.001785426049$$
$$a_7 := 0.00007977392784$$
$$b_7 := -0.001120757742$$
$$a_8 := -0.00004538043066$$
$$b_8 := 0.0007469694637$$
$$a_9 := 0.00002784809355$$
$$b_9 := -0.0005204913884$$
$$a_{10} := -0.00001807588500$$
$$b_{10} := 0.0003750000270$$
$$a_{11} := 0.00001229831819$$
$$b_{11} := -0.0002769549575$$
$$a_{12} := -0.000008712583672$$
$$b_{12} := 0.0002081358917$$
$$a_{13} := 0.000006401758364$$
$$b_{13} := -0.0001580479989$$
$$a_{14} := -0.000004869236979$$
$$b_{14} := 0.0001203387410$$
$$a_{15} := 0.000003834199040$$
$$b_{15} := -0.00009099027176$$
$$a_{16} := -0.000003130130832$$
$$b_{16} := 0.00006735418296$$

$$a_{17} := 0.000002656964690$$

$$b_{17} := -0.00004760930360$$

$$a_{18} := -0.000002354317186$$

$$b_{18} := 0.00003044206095$$

$$a_{19} := 0.000002185304028$$

$$b_{19} := -0.00001484799162$$

$$a_{20} := -0.000002130808000$$

$$b_{20} := 5.637851297\ 10^{-19}$$

We now form the trigonometric polynomial over the interval [-Pi, Pi] in the variable z. We name this polynomial s.

```
> s:=z->a[0]/2+a[m]/2*cos(m*z)+sum('a[k]*cos(k*z)+b[k]*sin(k*z)',
  'k'=1..m-1);
> s:=unapply(s(z),z);;
```

$$s := z \to \frac{1}{2}\, a_0 + \frac{1}{2}\, a_m \cos(m\, z) + \sum_{'k'=1}^{m-1} 'a_k \cos(k\, z) + b_k \sin(k\, z)'$$

$$\begin{aligned} s := z \to & -0.0005687667748 \cos(4\, z) + 0.7717267078 \cos(z) - 0.3870174297 \sin(z) \\ & + 0.01747923756 \cos(2\, z) + 0.04837532270 \sin(2\, z) + 0.006478440403 \cos(3\, z) \\ & - 0.01433101752 \sin(3\, z) + 0.7620272165 + 0.00001229831819 \cos(11\, z) \\ & - 0.0002769549575 \sin(11\, z) - 0.000008712583672 \cos(12\, z) + 0.0002081358917 \sin(12\, z) \\ & + 0.000006401758364 \cos(13\, z) - 0.0001580479989 \sin(13\, z) \\ & - 0.000004869236979 \cos(14\, z) + 0.0001203387410 \sin(14\, z) \\ & + 0.000003834199040 \cos(15\, z) - 0.00009099027176 \sin(15\, z) \\ & - 0.000003130130832 \cos(16\, z) + 0.00006735418296 \sin(16\, z) \\ & - 0.000002354317186 \cos(18\, z) + 0.00003044206095 \sin(18\, z) \\ & + 0.000002185304028 \cos(19\, z) - 0.00001484799162 \sin(19\, z) \\ & + 0.000002656964690 \cos(17\, z) - 0.00004760930360 \sin(17\, z) \\ & - 0.000001065404000 \cos(20\, z) + 0.006043110066 \sin(4\, z) + 0.0003637157810 \cos(5\, z) \\ & - 0.003090990282 \sin(5\, z) - 0.0001502368434 \cos(6\, z) + 0.001785426049 \sin(6\, z) \\ & + 0.00007977392784 \cos(7\, z) - 0.001120757742 \sin(7\, z) - 0.00004538043066 \cos(8\, z) \\ & + 0.0007469694637 \sin(8\, z) + 0.00002784809355 \cos(9\, z) - 0.0005204913884 \sin(9\, z) \\ & - 0.00001807588500 \cos(10\, z) + 0.0003750000270 \sin(10\, z) \end{aligned}$$

We translate the trigonometric polynomial back to our original interval [A, B].

```
> Z:=rhs(eq);
> S:=unapply(s(Z),x);
```

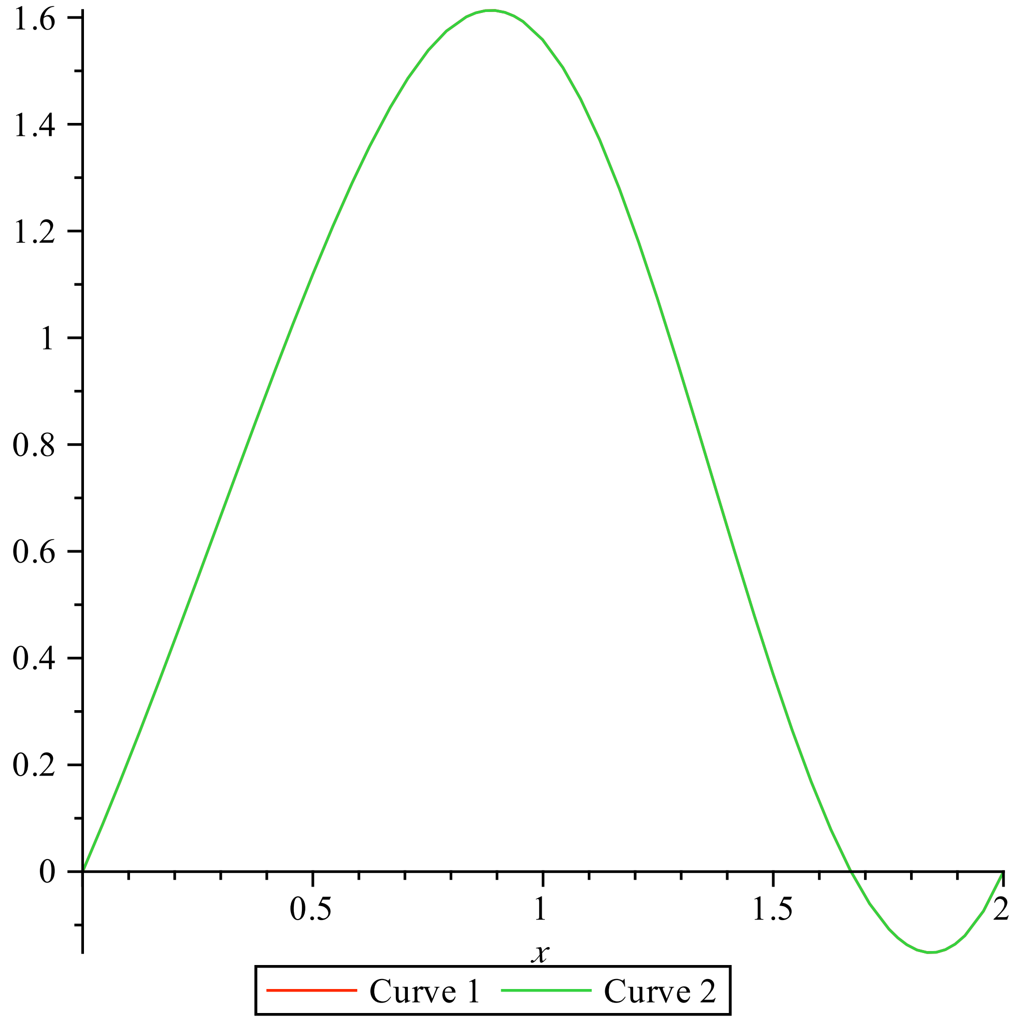
$$Z := -\pi + \pi\, x$$

$$\begin{aligned} S := x \to & -0.0005687667748 \cos(4\, \pi\, x) - 0.7717267078 \cos(\pi\, x) + 0.3870174297 \sin(\pi\, x) \\ & + 0.01747923756 \cos(2\, \pi\, x) + 0.04837532270 \sin(2\, \pi\, x) - 0.006478440403 \cos(3\, \pi\, x) \end{aligned}$$

$+ 0.01433101752 \sin(3 \pi x) + 0.7620272165 - 0.00001229831819 \cos(11 \pi x)$
$+ 0.0002769549575 \sin(11 \pi x) - 0.000008712583672 \cos(12 \pi x)$
$+ 0.0002081358917 \sin(12 \pi x) - 0.000006401758364 \cos(13 \pi x)$
$+ 0.0001580479989 \sin(13 \pi x) - 0.000004869236979 \cos(14 \pi x)$
$+ 0.0001203387410 \sin(14 \pi x) - 0.000003834199040 \cos(15 \pi x)$
$+ 0.00009099027176 \sin(15 \pi x) - 0.000003130130832 \cos(16 \pi x)$
$+ 0.00006735418296 \sin(16 \pi x) - 0.000002354317186 \cos(18 \pi x)$
$+ 0.00003044206095 \sin(18 \pi x) - 0.000002185304028 \cos(19 \pi x)$
$+ 0.00001484799162 \sin(19 \pi x) - 0.000002656964690 \cos(17 \pi x)$
$+ 0.00004760930360 \sin(17 \pi x) - 0.000001065404000 \cos(20 \pi x)$
$+ 0.006043110066 \sin(4 \pi x) - 0.0003637157810 \cos(5 \pi x) + 0.003090990282 \sin(5 \pi x)$
$- 0.0001502368434 \cos(6 \pi x) + 0.001785426049 \sin(6 \pi x)$
$- 0.00007977392784 \cos(7 \pi x) + 0.001120757742 \sin(7 \pi x)$
$- 0.00004538043066 \cos(8 \pi x) + 0.0007469694637 \sin(8 \pi x)$
$- 0.00002784809355 \cos(9 \pi x) + 0.0005204913884 \sin(9 \pi x)$
$- 0.00001807588500 \cos(10 \pi x) + 0.0003750000270 \sin(10 \pi x)$

We graph the function (red) and the trigonometric polynomial (green) on the interval [A, B].

```
> plot({f(x),S(x)},x=A..B);
```



This looks very good. We can't distinguish between the two graphs.