

Polynomial Roots

Newton's Method with Horner's Method

nalib

We wish to find all four roots of the polynomial

$$p(x) = x^4 - 2x^3 - 12x^2 + 16x - 40,$$

both real and complex. We graph the polynomial.

```
> restart;
```

```
> libname:="c:/nalib", libname;
```

```
libname := "/nalib", "/Library/Frameworks/Maple.framework/Versions/15/lib",
```

```
"/Library/Frameworks/Maple.framework/Versions/15/toolbox/NAG/lib"
```

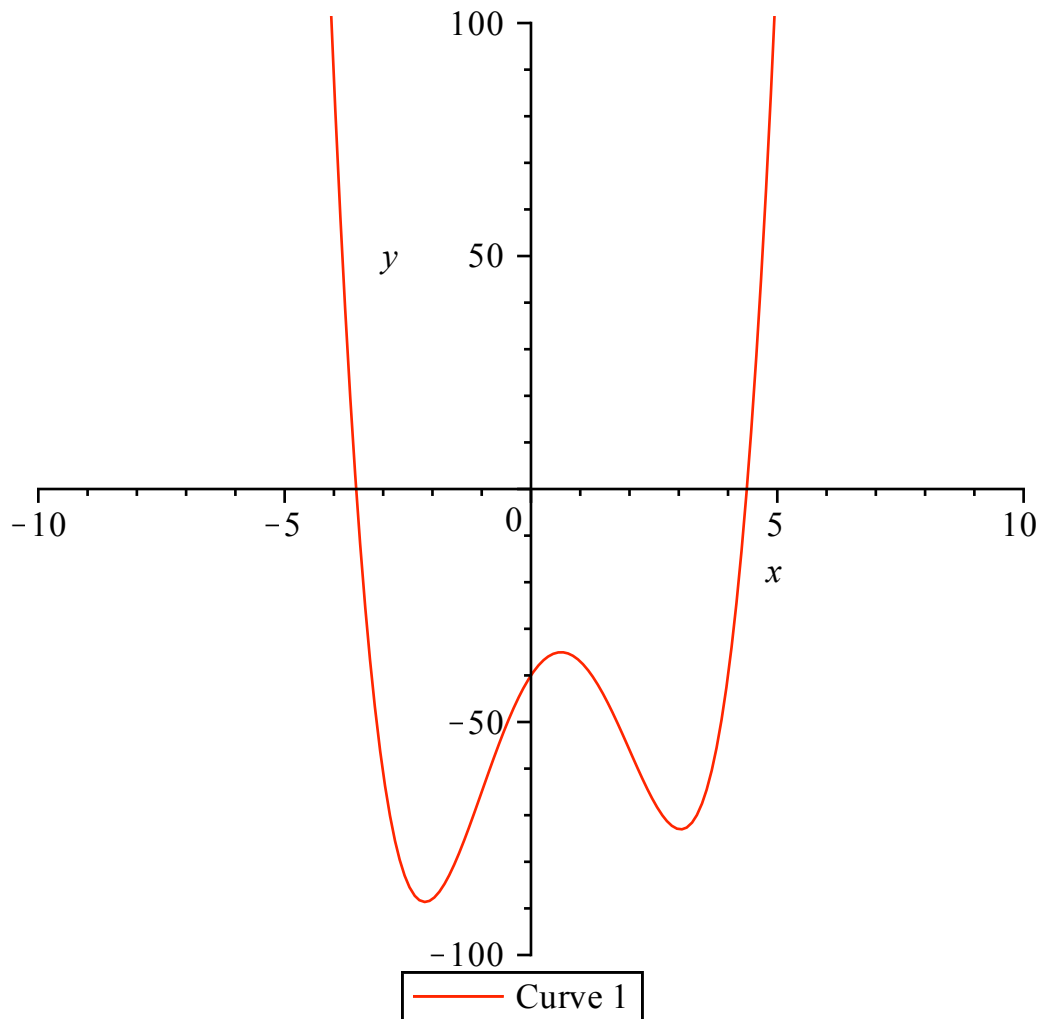
```
> with(numanal);
```

```
[SOR, SOR_dir, adaptq, adaptq_dir, bezier, bezier_dir, bisection, bisection_dir, chop, chop_dir,
clamped_spline, clamped_spline_dir, divided_diff, divided_diff_dir, extrap, extrap_dir,
falseposition, falseposition_dir, fixedpoint, fixedpoint_dir, gaussseidel, gaussseidel_dir, hermite,
hermite_dd, hermite_dd_dir, hermite_dir, horner, horner_dir, jacobi, jacobi_dir, muller,
muller_dir, natural_spline, natural_spline_dir, newton, newton_dir, romberg, romberg_dir,
secant, secant_dir, steffensen, steffensen_dir]
```

```
> p:=x^4-2*x^3-12*x^2+16*x-40;
```

$$p := x^4 - 2x^3 - 12x^2 + 16x - 40$$

```
> plot(p, x=-10..10, y=-100..100);
```



We first look for the root near 4. Suppose we wish to find the roots within 10^{-6} . We use **newton**.

```
> r1:=newton(p,4,.000001,100,r1);
```

i	p	f(p)
0	4	-40.
1	4.500000000	16.81250000
2	4.388658940	1.00080784
3	4.381146397	.435205e-2
4	4.381113442	.7e-7
5	4.381113441	-.4e-7

The approximate solution is $r1 = 4.381113441$
with $f(r1) = -.4e-7$

```
r1 := 4.381113441
```

Before finding the next root, we **deflate** the polynomial by using Horner's method, implemented by **horner**. We first check the directions.

```
> horner_dir();
```

horner returns a list where the first entry is the quotient polynomial, the second is the value of the original polynomial at the point of evaluation, and the third is the value of the derivative of the original polynomial at the point of evaluation.

The arguments for horner are, in order:

- (1) the degree of the polynomial
- (2) the point of evaluation (real or complex)
- (3) the coefficients (real or complex) in descending order by degree

```
> h1:=horner(4,r1,coeff(p,x,4),coeff(p,x,3),coeff(p,x,2),coeff(p,x,1),coeff(p,x,0));
```

$$h1 := [x^3 + 2.381113441 x^2 - 1.56807190 x + 9.130099122, -2.10^{-8}, 132.0554290]$$

We set the deflated polynomial equal to $q1$. We treat -2.10^{-8} as 0.

```
> q1:=h1[1];
```

$$q1 := x^3 + 2.381113441 x^2 - 1.56807190 x + 9.130099122$$

Now we find a root of this polynomial using **newton**. Using the graph, we'll start at -4.

```
> r2:=newton(q1,-4,.000001,100,r2);
```

i	p	f(p)
-	-	----
0	-4	-10.49979822
1	-3.616558071	-1.357866288
2	-3.550150736	-.37052928e-1
3	-3.548234471	-.30369e-4
4	-3.548232898	.4e-8
5	-3.548232898	.4e-8

The approximate solution is $r2 = -3.548232898$
with $f(r2) = .4e-8$

$$r2 := -3.548232898$$

Again, we deflate this polynomial by Horner's method.

```
> h2:=horner(3,r2,coeff(q1,x,3),coeff(q1,x,2),coeff(q1,x,1),coeff(q1,x,0));
```

$$h2 := [x^2 - 1.167119457 x + 2.573139753, -1.10^{-9}, 19.30430810]$$

We set this deflated polynomial equal to $q2$. Again, we treat -1.10^{-9} as 0;

```
> q2:=h2[1];
```

$$q2 := x^2 - 1.167119457 x + 2.573139753$$

We find one of the roots of this equation by using the quadratic formula.

```
> r3:=(-coeff(q2,x,1)+sqrt(coeff(q2,x,1)^2-4*coeff(q2,x,2)*coeff(q2,x,0)))/(2*coeff(q2,x,2));
```

$$r3 := 0.5835597285 + 1.494188006 I$$

Now we tighten up each root by using them as first approximations in **newton** in the original function.

```
> x1:=newton(p,r1,.000001,100,x1);
```

i	p	f(p)
-	-	----
0	4.381113441	-.4e-7
1	4.381113441	-.4e-7

The approximate solution is $x1 = 4.381113441$
with $f(x1) = -.4e-7$

$$x1 := 4.381113441$$

```
> x2:=newton(p,r2,.000001,100,x2);
```

i	p	f(p)
-	-	----
0	-3.548232898	.3e-7
1	-3.548232898	.3e-7

The approximate solution is $x_2 = -3.548232898$
with $f(x_2) = .3e-7$

$x_2 := -3.548232898$

> x3:=newton(p,r3,.000001,100,x3);

i	p	f(p)
0	.5835597285+1.494188006*I	-.4e-8+0.*I
1	.5835597285+1.494188006*I	-.4e-8+0.*I

The approximate solution is $x_3 = .5835597285+1.494188006*I$
with $f(x_3) = -.4e-8+0.*I$

$x_3 := 0.5835597285 + 1.494188006 I$

For polynomials with real coefficients, complex roots come in conjugate pairs. So for the fourth root, we find the conjugate of the third.

> x4:=conjugate(x3);

$x_4 := 0.5835597285 - 1.494188006 I$