```
>
```

Fourier Series

Period 2π over the interval $[-\pi, \pi]$.

```
>
> restart:with(plots):
```

Warning, the name changecoords has been redefined

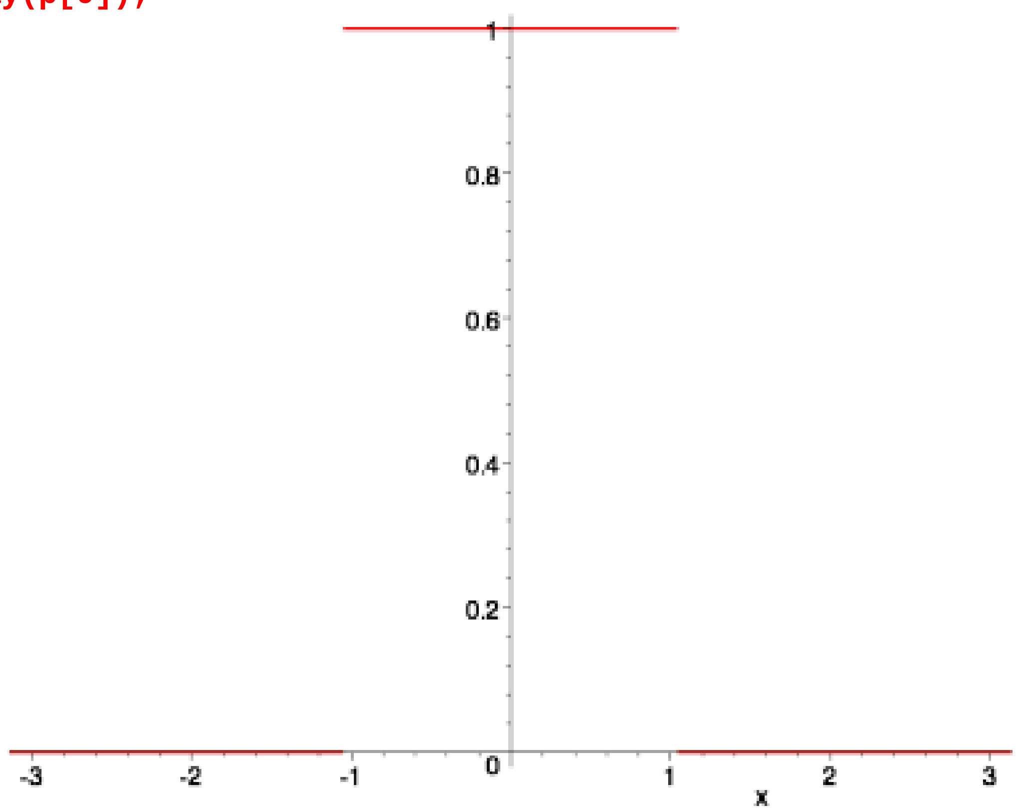
We begin by considering the function

```
> f:=piecewise(x<-Pi/3,0,x<Pi/3,1,0);
```

$$f := \begin{cases} 0 & x < -\frac{1}{3}\pi \\ 1 & x < \frac{1}{3}\pi \\ 0 & otherwise \end{cases}$$

over the interval x = $-\pi..\pi$, whose plot is below.

```
> p[0]:=plot(f,x=-Pi..Pi,thickness=3,discont=true):
  display(p[0]);
```



We will approximate this function by Fourier polynomials of degrees 1 through 15.

```
> n:=15;
```

$$n := 15$$

We first compute the coefficient $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$.

```
> a[0]:=(1/(2*Pi))*int(f,x=-Pi..Pi);
```

$$a_0 := \frac{1}{3}$$

Next we use a loop to compute the coefficients $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(k\,x)\, dx$ and $b_k = \frac{1}{\pi}$ $\int_{-\pi}^{\pi} f(x) \sin(k\,x)\, dx$, along with the Fourier polynomials

$F_n(x) = a_0 + \left(\sum_{k=1}^{n} a_k \cos(k\,x)\right) + \left(\sum_{k=1}^{n} b_k \sin(k\,x)\right)$ for n = 1..15.

```
> for k from 1 to n do
  a[k]:=(1/Pi)*int(f*cos(k*x),x=-Pi..Pi);
  b[k]:=(1/Pi)*int(f*sin(k*x),x=-Pi..Pi);
  F[k]:=a[0]+sum('a[i]*cos(i*x)','i'=1..k)+sum('b[i]*sin(i*x)','i'=1..k
  );
  od;
```

$$a_1 := \frac{\sqrt{3}}{\pi}$$

$$b_1 := 0$$

$$F_1 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi}$$

$$a_2 := \frac{1}{2}\frac{\sqrt{3}}{\pi}$$

$$b_2 := 0$$

$$F_2 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2\,x)}{\pi}$$

$$a_3 := 0$$

$$b_3 := 0$$

$$F_3 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2\,x)}{\pi}$$

$$a_4 := -\frac{1}{4}\frac{\sqrt{3}}{\pi}$$

$$b_4 := 0$$

$$F_4 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2\,x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4\,x)}{\pi}$$

$$a_5 := -\frac{1}{5}\frac{\sqrt{3}}{\pi}$$

$$b_5 := 0$$

$$F_5 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}$$

$$a_6 := 0$$

$$b_6 := 0$$

$$F_6 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}$$

$$a_7 := \frac{1}{7}\frac{\sqrt{3}}{\pi}$$

$$b_7 := 0$$

$$F_7 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$

$$a_8 := \frac{1}{8}\frac{\sqrt{3}}{\pi}$$

$$b_8 := 0$$

$$F_8 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi} + \frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}$$

$$a_9 := 0$$

$$b_9 := 0$$

$$F_9 := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi} + \frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}$$

$$a_{10} := -\frac{1}{10}\frac{\sqrt{3}}{\pi}$$

$$b_{10} := 0$$

$$F_{10} := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi} + \frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$

$$+\frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}-\frac{1}{10}\frac{\sqrt{3}\cos(10x)}{\pi}$$

$$a_{11} := -\frac{1}{11}\frac{\sqrt{3}}{\pi}$$

$$b_{11} := 0$$

$$F_{11} := \frac{1}{3}+\frac{\sqrt{3}\cos(x)}{\pi}+\frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi}-\frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi}-\frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}+\frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$
$$+\frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}-\frac{1}{10}\frac{\sqrt{3}\cos(10x)}{\pi}-\frac{1}{11}\frac{\sqrt{3}\cos(11x)}{\pi}$$

$$a_{12} := 0$$

$$b_{12} := 0$$

$$F_{12} := \frac{1}{3}+\frac{\sqrt{3}\cos(x)}{\pi}+\frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi}-\frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi}-\frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}+\frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$
$$+\frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}-\frac{1}{10}\frac{\sqrt{3}\cos(10x)}{\pi}-\frac{1}{11}\frac{\sqrt{3}\cos(11x)}{\pi}$$

$$a_{13} := \frac{1}{13}\frac{\sqrt{3}}{\pi}$$

$$b_{13} := 0$$

$$F_{13} := \frac{1}{3}+\frac{\sqrt{3}\cos(x)}{\pi}+\frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi}-\frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi}-\frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}+\frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$
$$+\frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}-\frac{1}{10}\frac{\sqrt{3}\cos(10x)}{\pi}-\frac{1}{11}\frac{\sqrt{3}\cos(11x)}{\pi}+\frac{\frac{1}{13}\sqrt{3}\cos(13x)}{\pi}$$

$$a_{14} := \frac{1}{14}\frac{\sqrt{3}}{\pi}$$

$$b_{14} := 0$$

$$F_{14} := \frac{1}{3}+\frac{\sqrt{3}\cos(x)}{\pi}+\frac{\frac{1}{2}\sqrt{3}\cos(2x)}{\pi}-\frac{1}{4}\frac{\sqrt{3}\cos(4x)}{\pi}-\frac{1}{5}\frac{\sqrt{3}\cos(5x)}{\pi}+\frac{\frac{1}{7}\sqrt{3}\cos(7x)}{\pi}$$
$$+\frac{\frac{1}{8}\sqrt{3}\cos(8x)}{\pi}-\frac{1}{10}\frac{\sqrt{3}\cos(10x)}{\pi}-\frac{1}{11}\frac{\sqrt{3}\cos(11x)}{\pi}+\frac{\frac{1}{13}\sqrt{3}\cos(13x)}{\pi}$$

$$+\frac{\frac{1}{14}\sqrt{3}\cos(14\,x)}{\pi}$$

$$a_{15} := 0$$

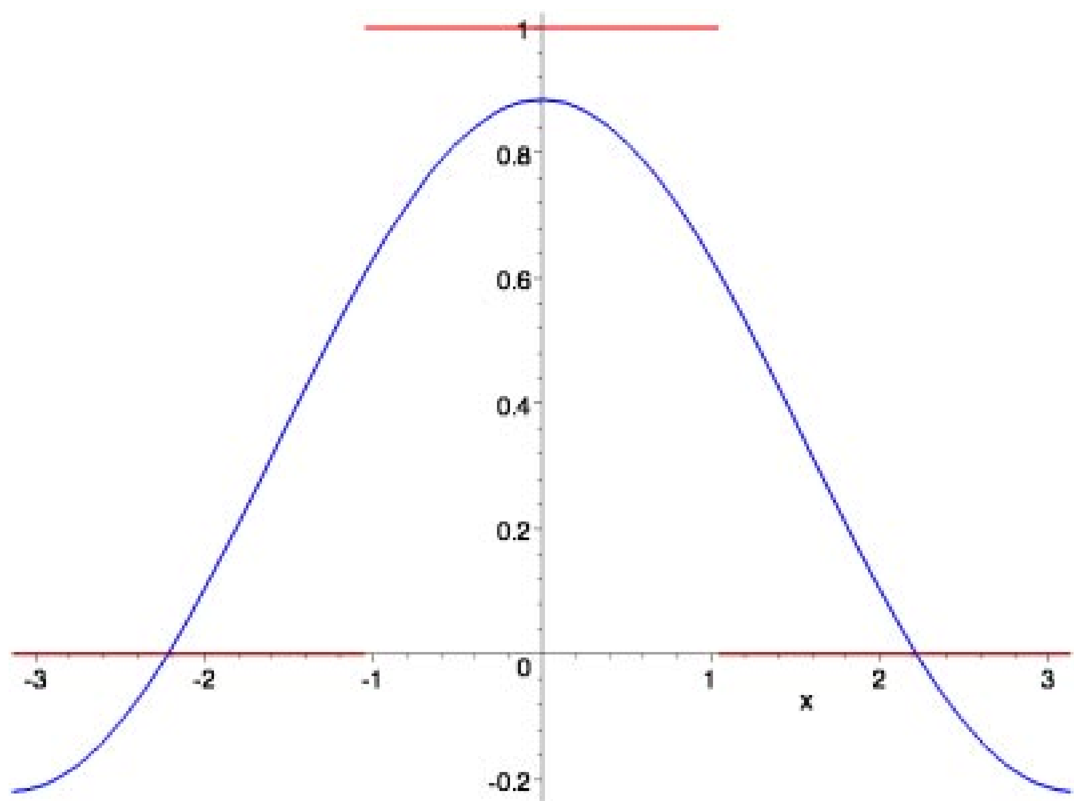
$$b_{15} := 0$$

$$F_{15} := \frac{1}{3} + \frac{\sqrt{3}\cos(x)}{\pi} + \frac{\frac{1}{2}\sqrt{3}\cos(2\,x)}{\pi} - \frac{1}{4}\frac{\sqrt{3}\cos(4\,x)}{\pi} - \frac{1}{5}\frac{\sqrt{3}\cos(5\,x)}{\pi} + \frac{\frac{1}{7}\sqrt{3}\cos(7\,x)}{\pi}$$
$$+ \frac{\frac{1}{8}\sqrt{3}\cos(8\,x)}{\pi} - \frac{1}{10}\frac{\sqrt{3}\cos(10\,x)}{\pi} - \frac{1}{11}\frac{\sqrt{3}\cos(11\,x)}{\pi} + \frac{\frac{1}{13}\sqrt{3}\cos(13\,x)}{\pi}$$
$$+ \frac{\frac{1}{14}\sqrt{3}\cos(14\,x)}{\pi}$$

Next we plot f(x) along with each of the $F_n(x)$ for n = 1..15.

```
> for k from 1 to n do
  p[k]:=plot(F[k],x=-Pi..Pi,thickness=3,discont=true,color=blue):
  od:
> for k from 1 to n do
  k;
  display(p[0],p[k]):
  od;
```

1



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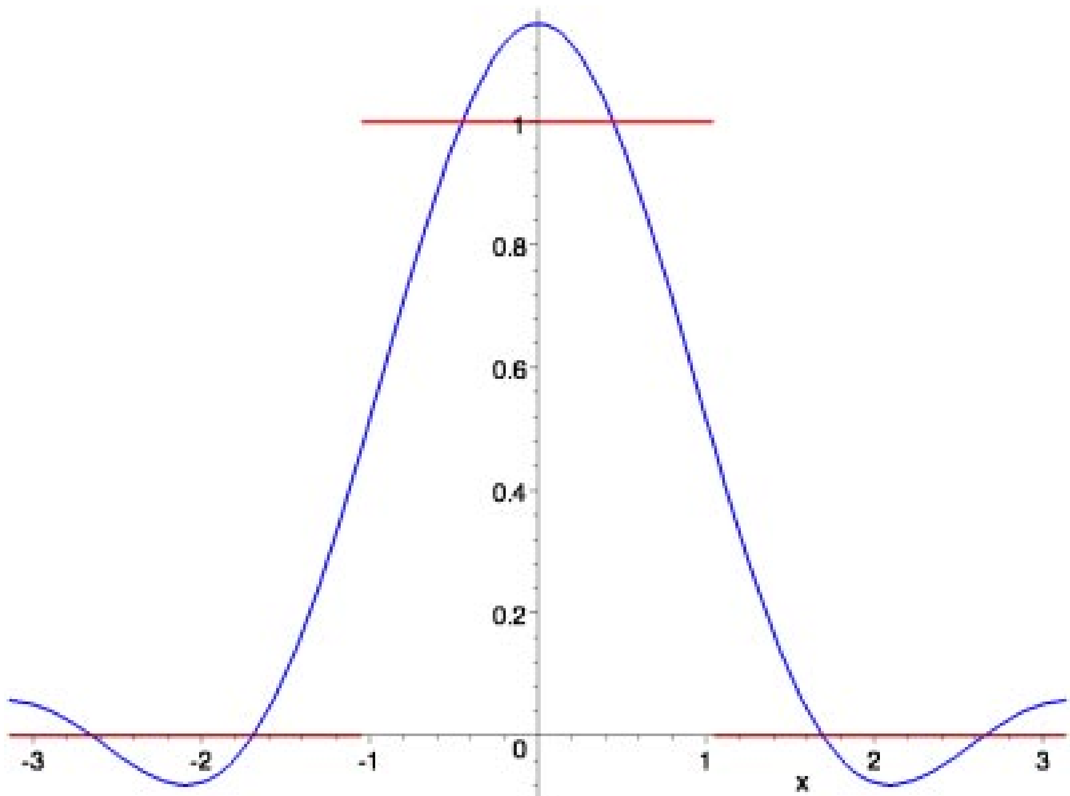

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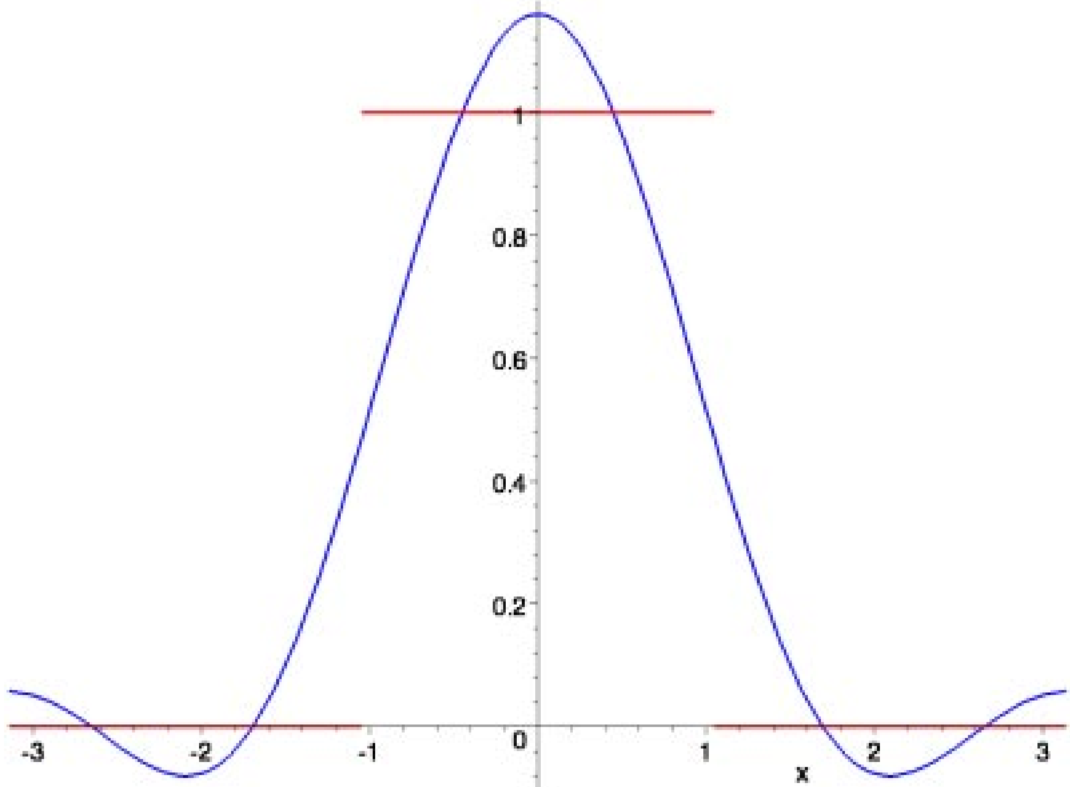

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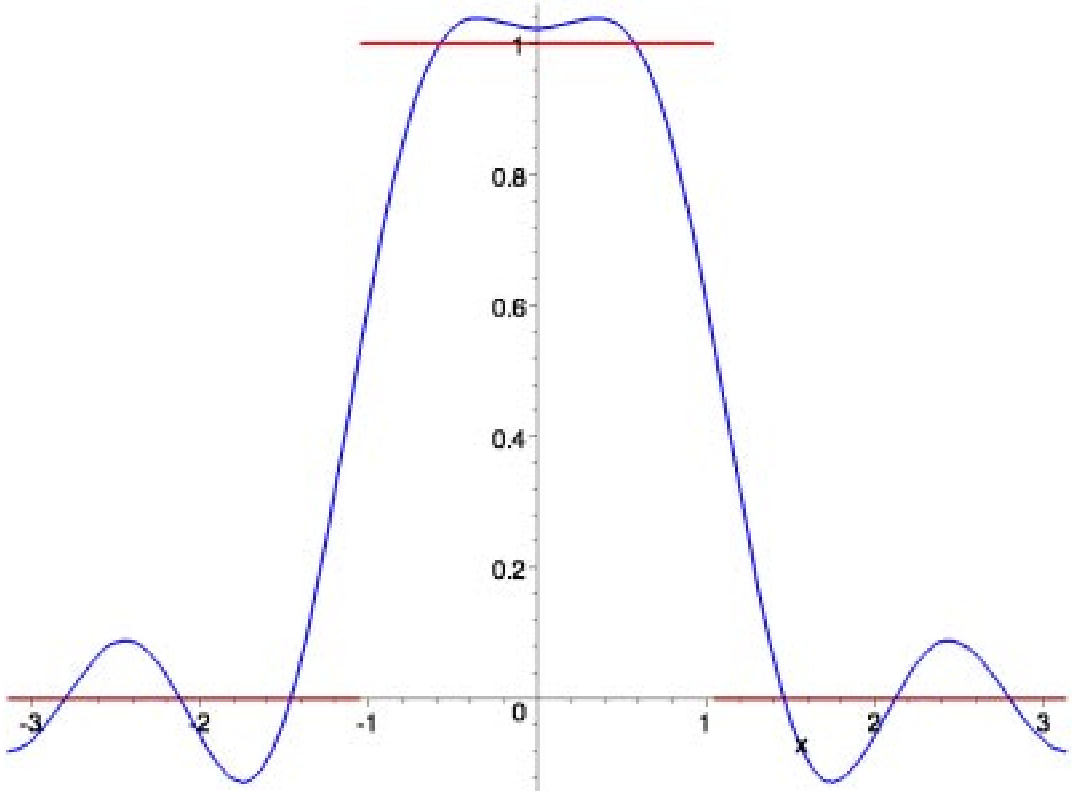

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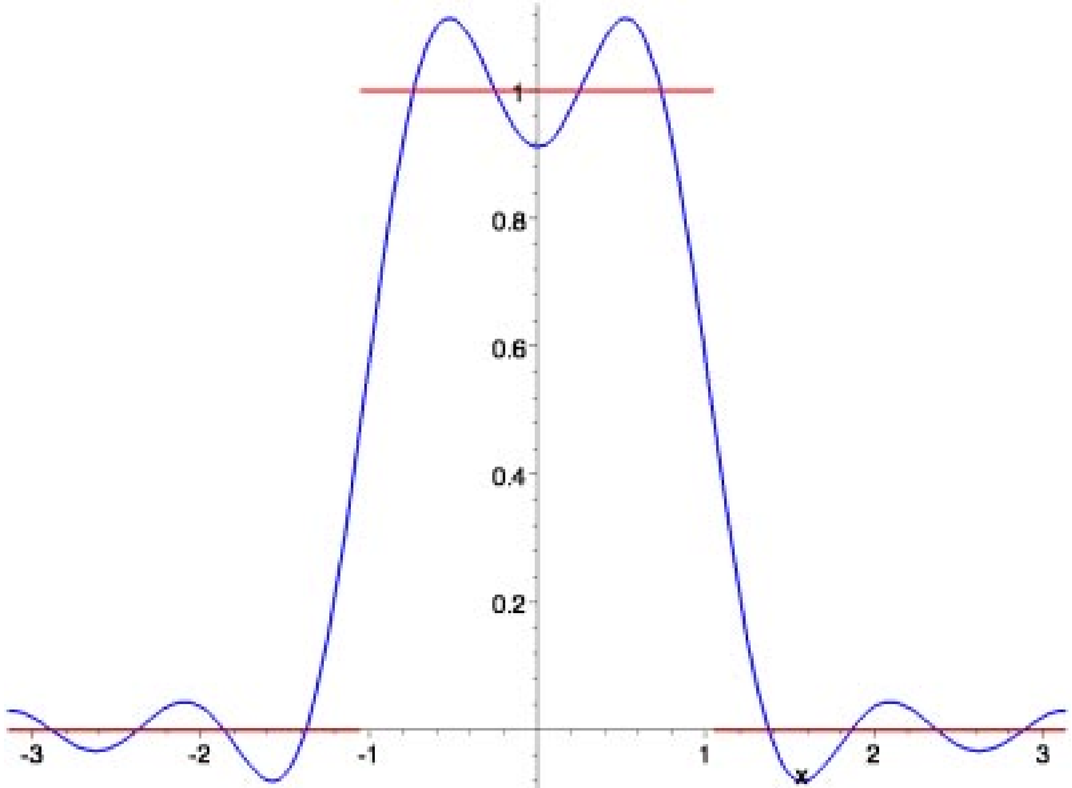

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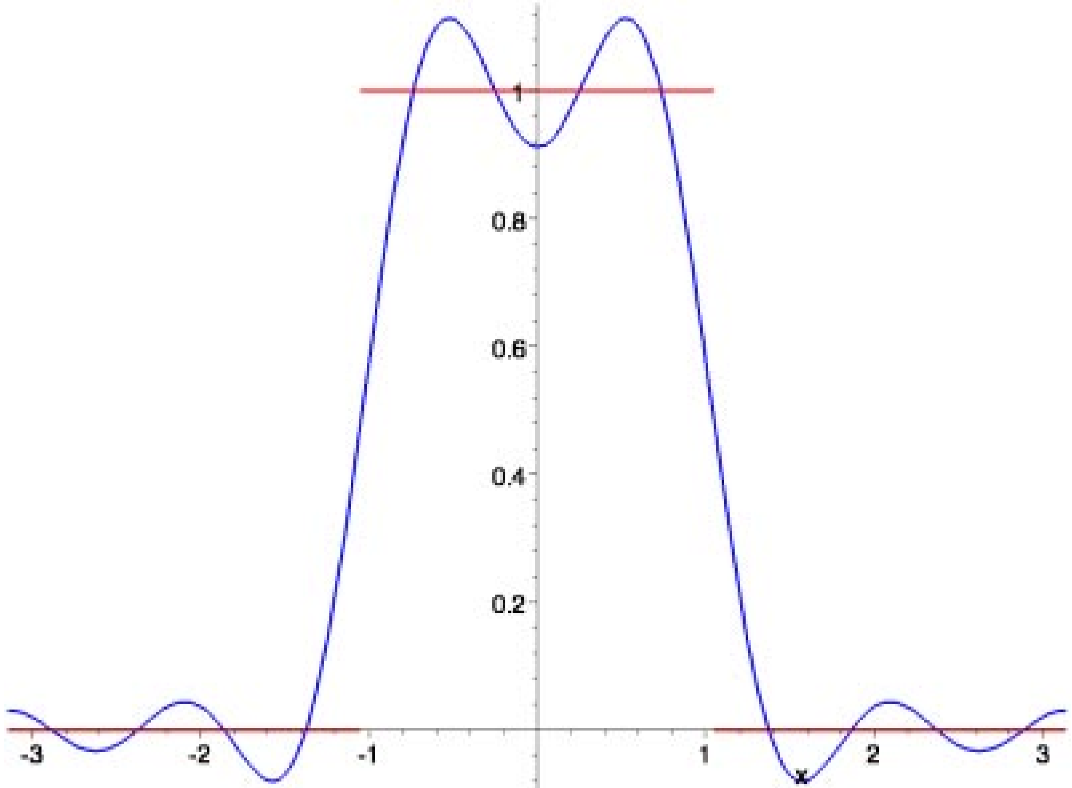

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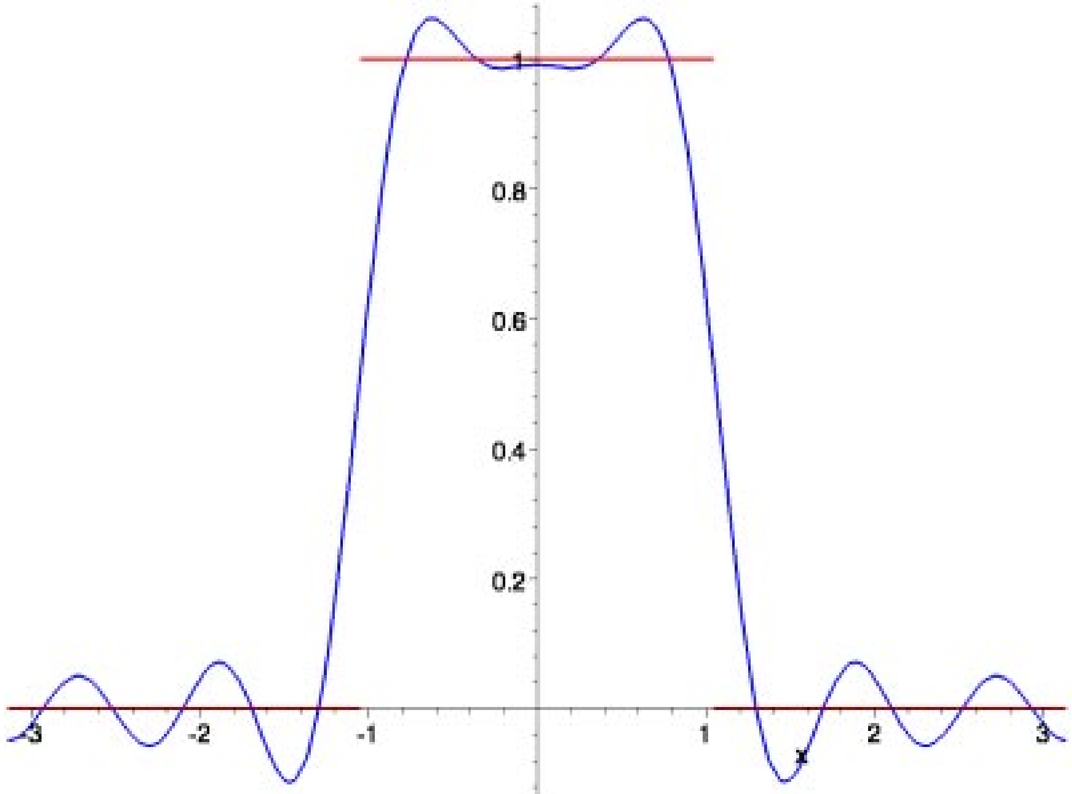

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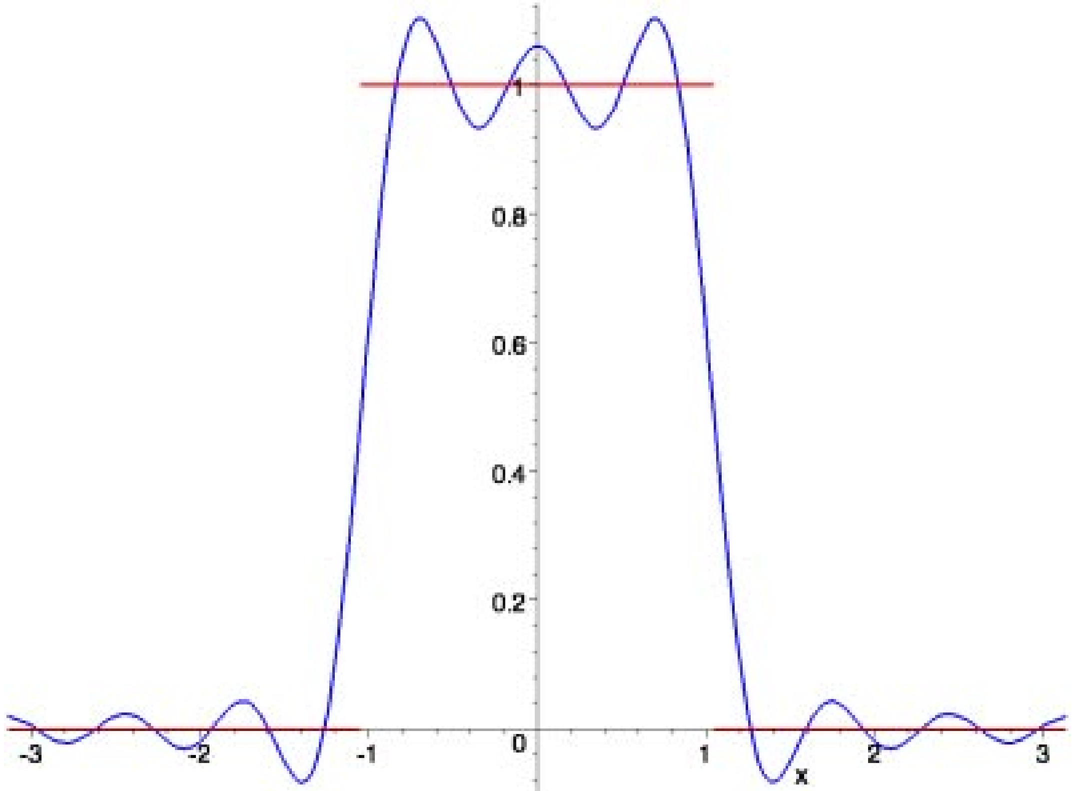

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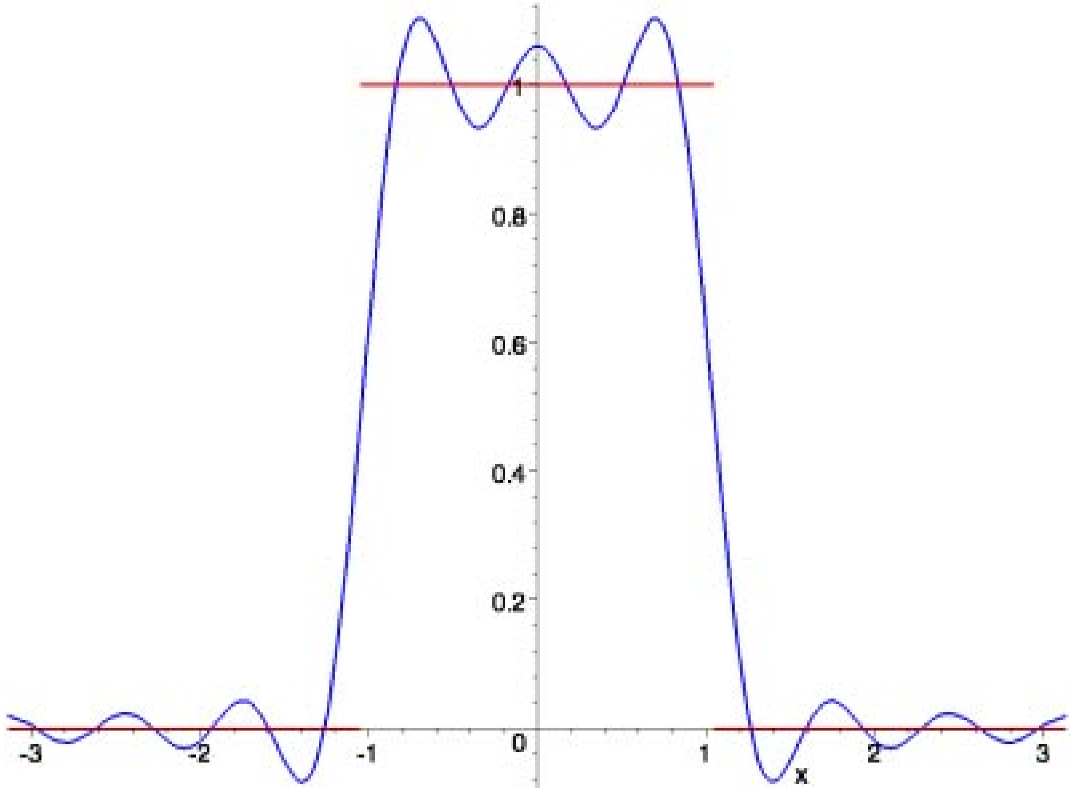

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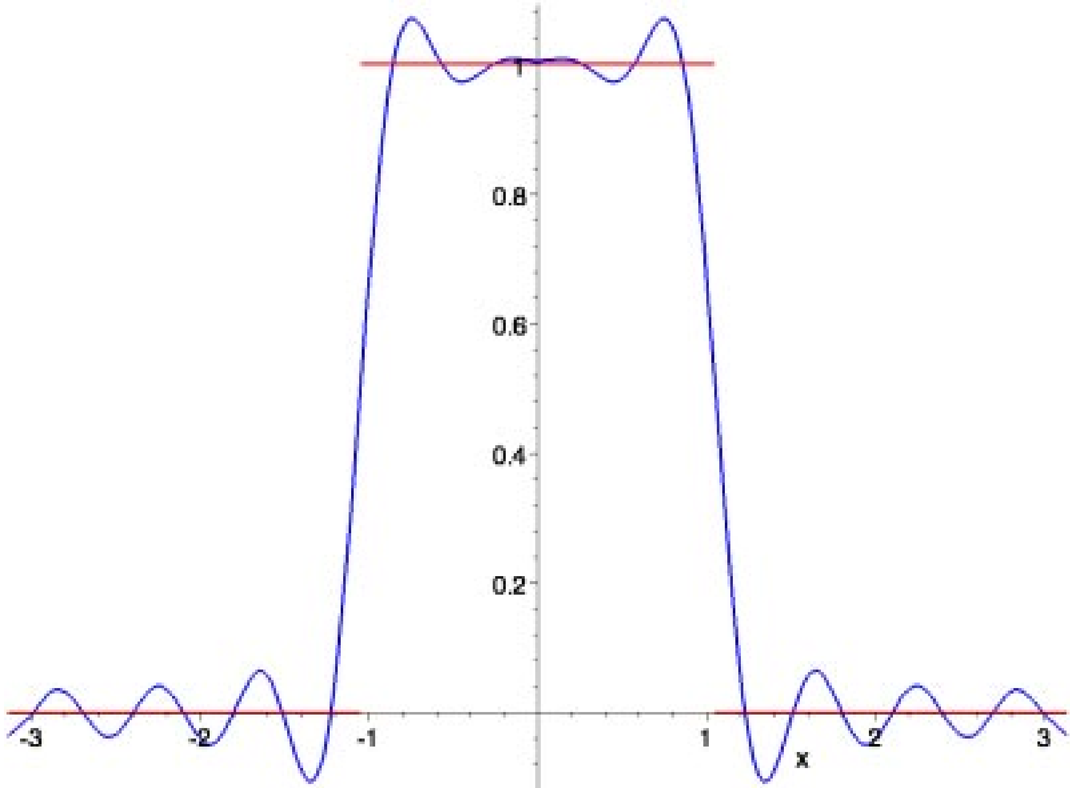

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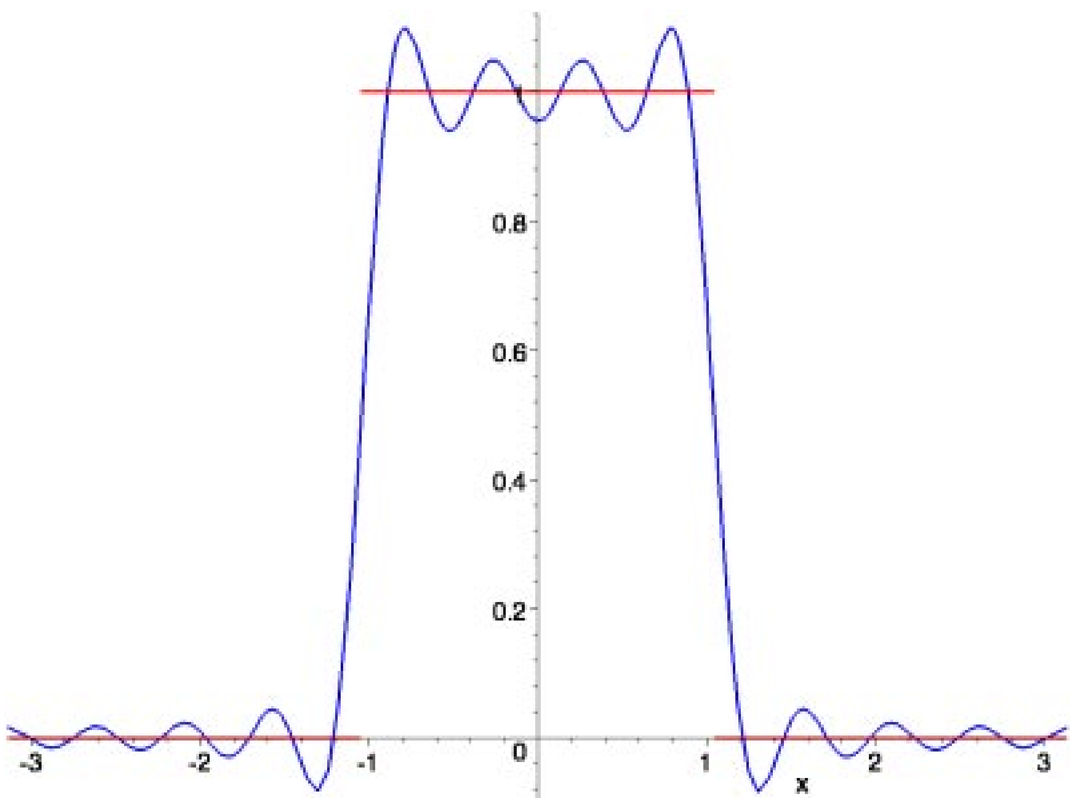

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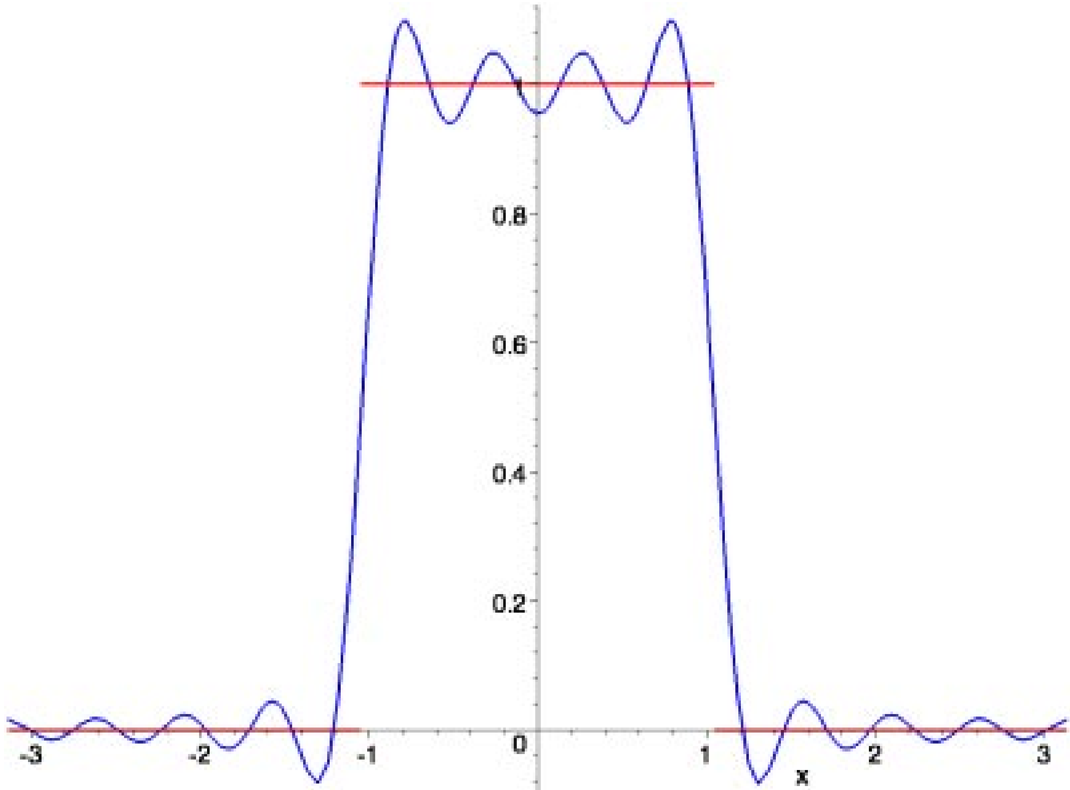

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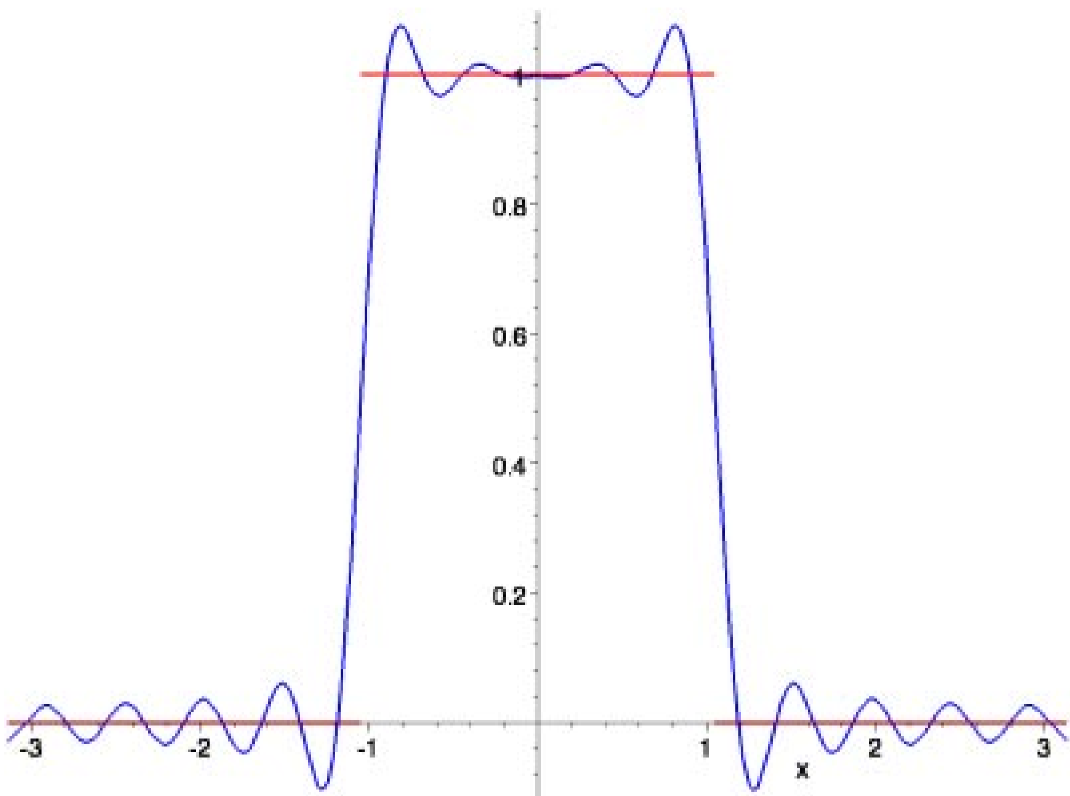

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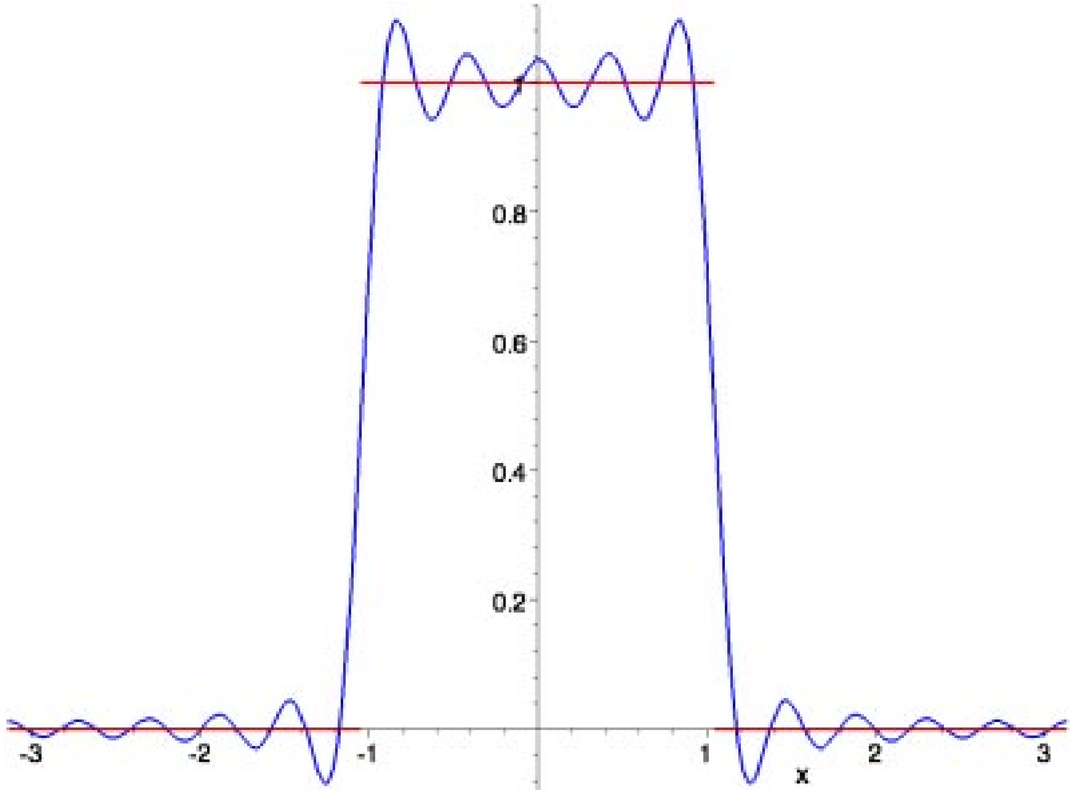

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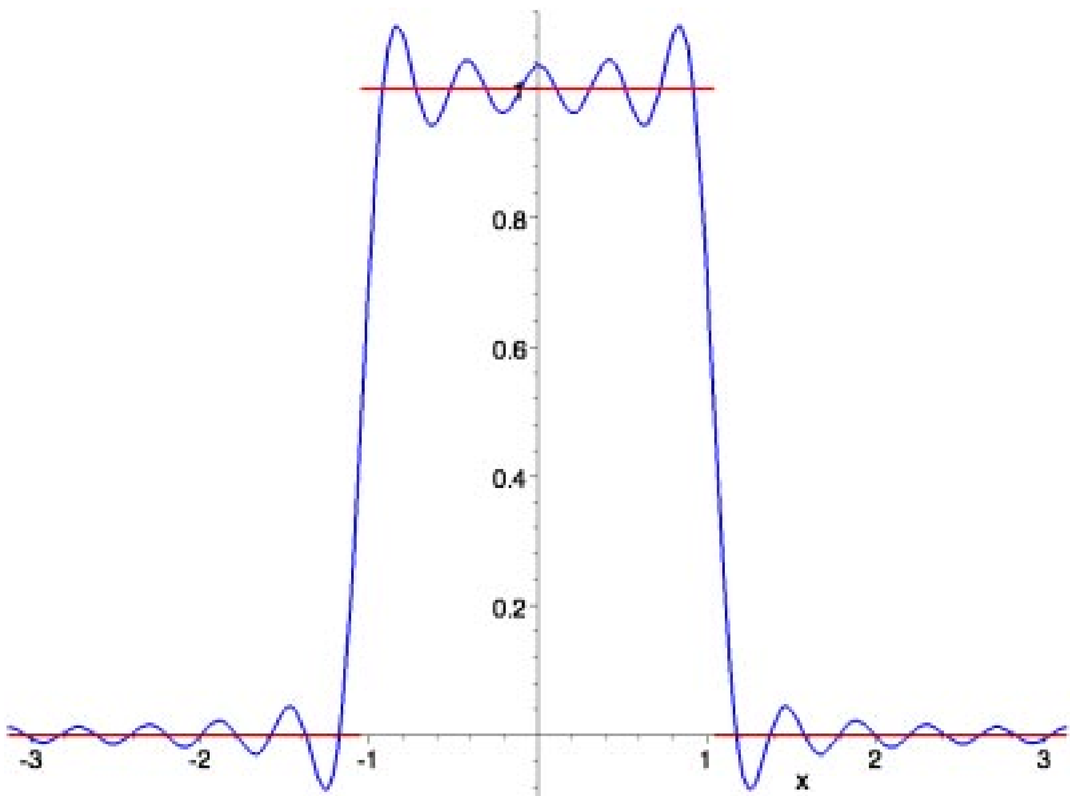


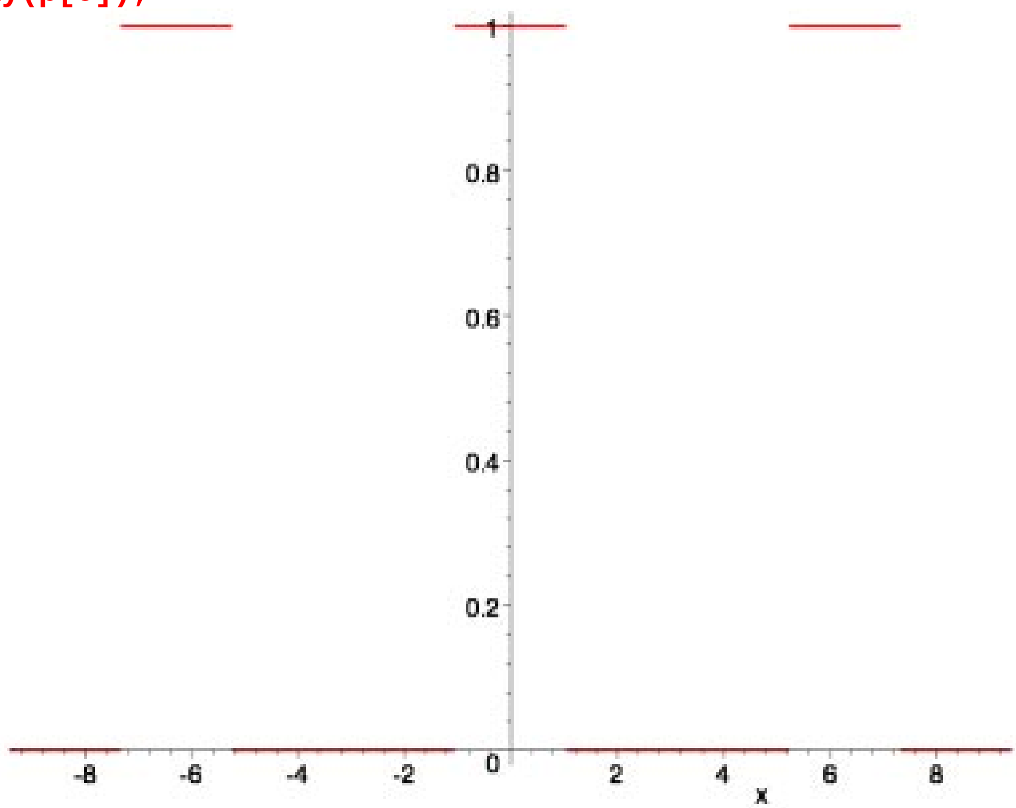
We see that the successive Fourier polynomials give better and better approximations to f(x). Although they may not be as accurate as Taylor polynomials near a given point, they do a better job of approximating a function over an entire interval. They are especially good for approximating periodic functions. Let's extend f(x) by a period in each direction with x = -3 π..3 π.

```
> f:=piecewise(x<-7*Pi/3,0,x<-5*Pi/3,1,x<-Pi/3,0,x<Pi/3,1,x<5*Pi/3,0,x<
  7*Pi/3,1,0);
```

$$f := \begin{cases} 0 & x < -\frac{7}{3}\pi \\ 1 & x < -\frac{5}{3}\pi \\ 0 & x < -\frac{1}{3}\pi \\ 1 & x < \frac{1}{3}\pi \\ 0 & x < \frac{5}{3}\pi \\ 1 & x < \frac{7}{3}\pi \\ 0 & \mathit{otherwise} \end{cases}$$

We graph the extention.

```
> p[0]:=plot(f,x=-3*Pi..3*Pi,thickness=3,discont=true):
  display(p[0]);
```

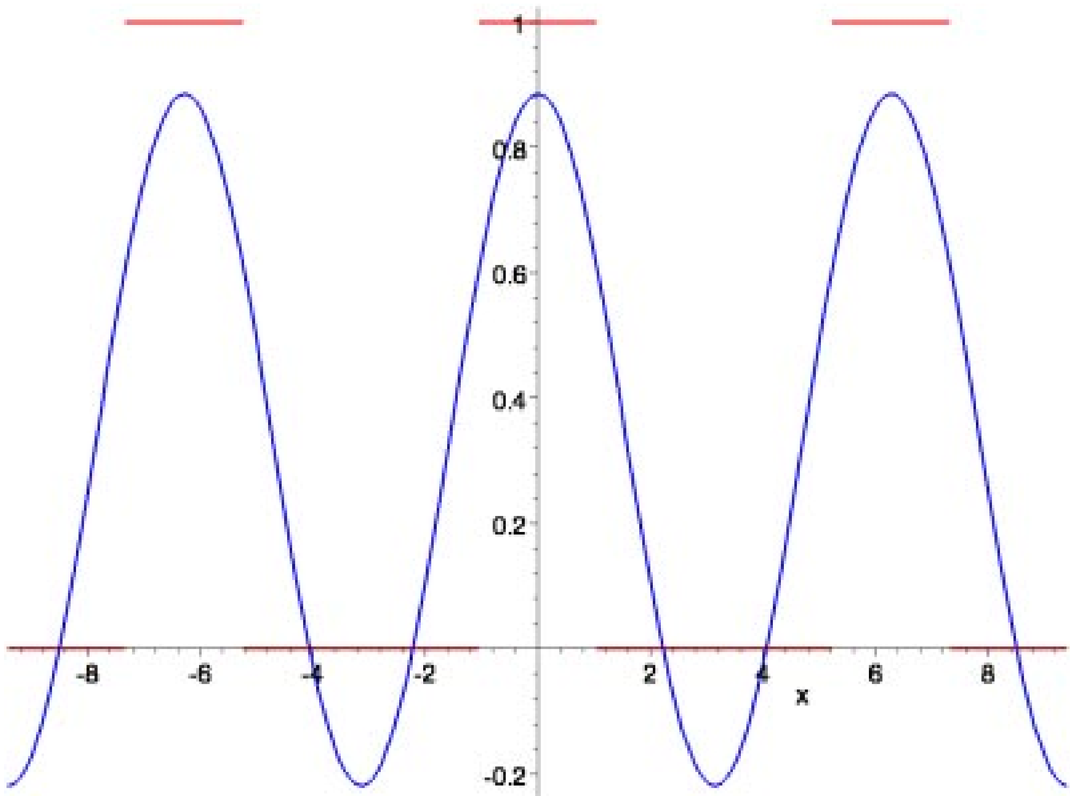


We now plot the extended f(x) along with each of the $F_n(x)$ for n = 1..15.

```
> for k from 1 to n do
  p[k]:=plot(F[k],x=-3*Pi..3*Pi,thickness=3,discont=true,color=blue):
  od:
> for k from 1 to n do
  k;
```

```
display(p[0],p[k]):
od;
```

1



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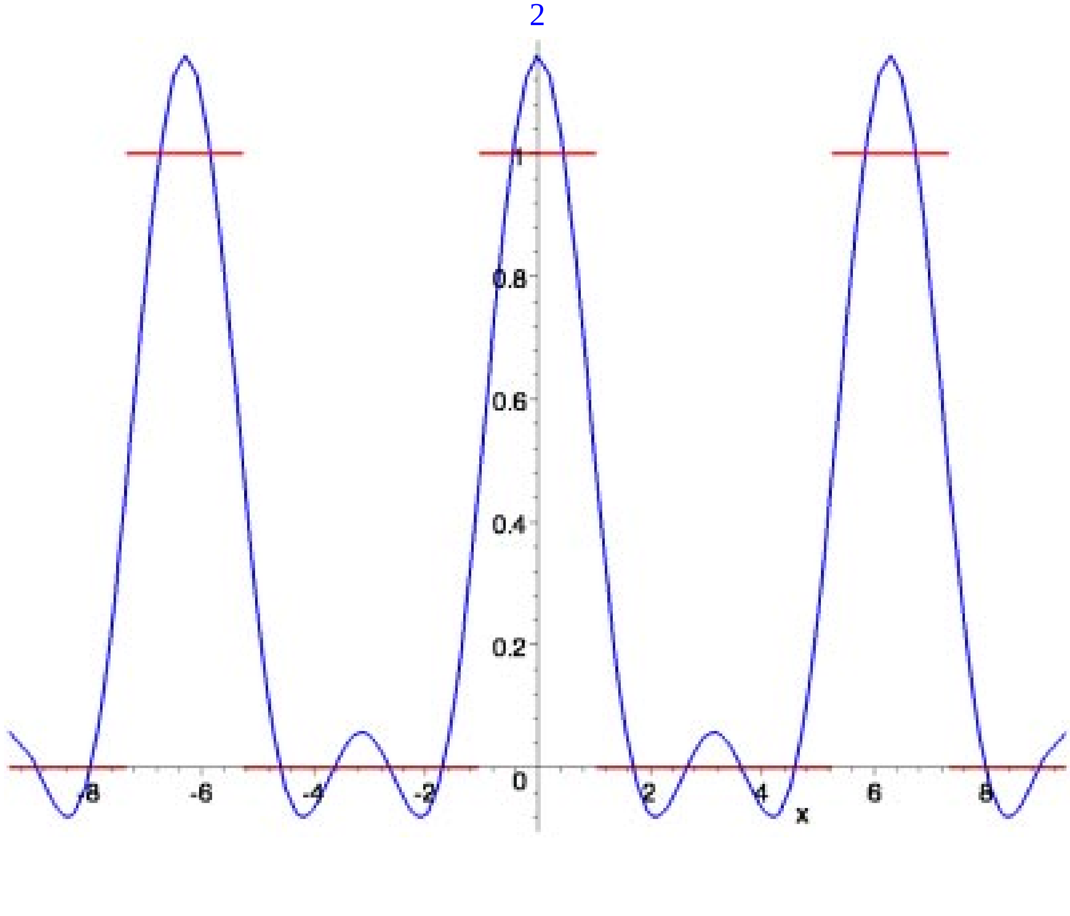

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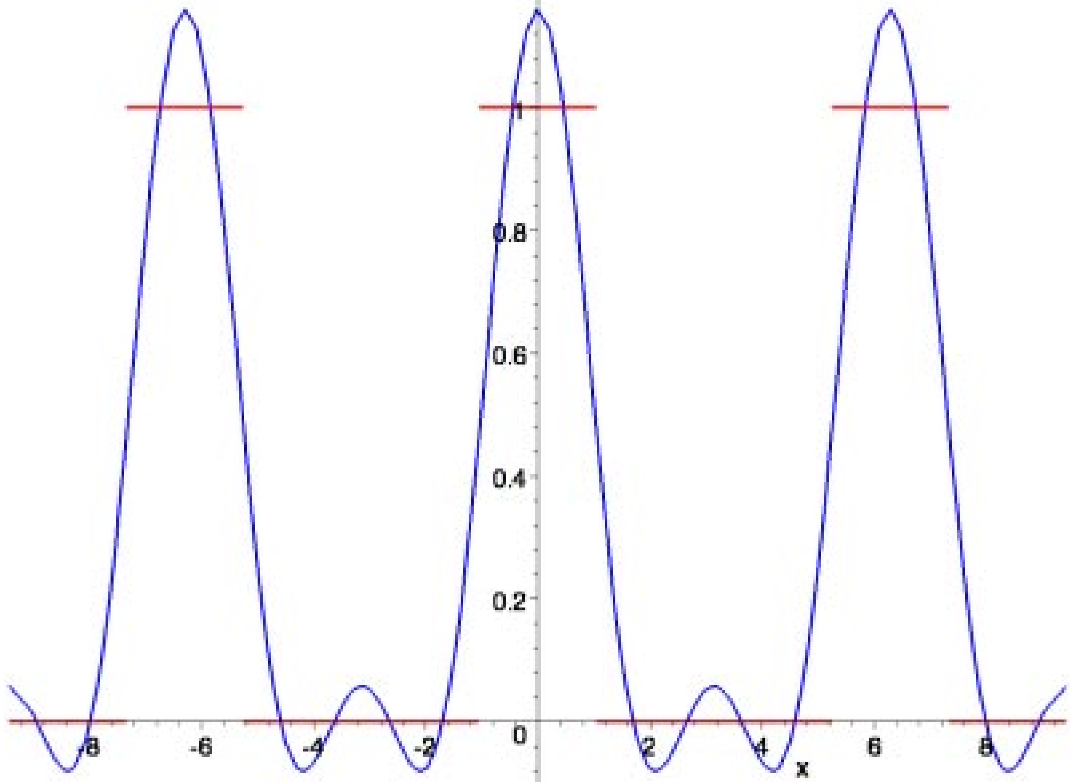

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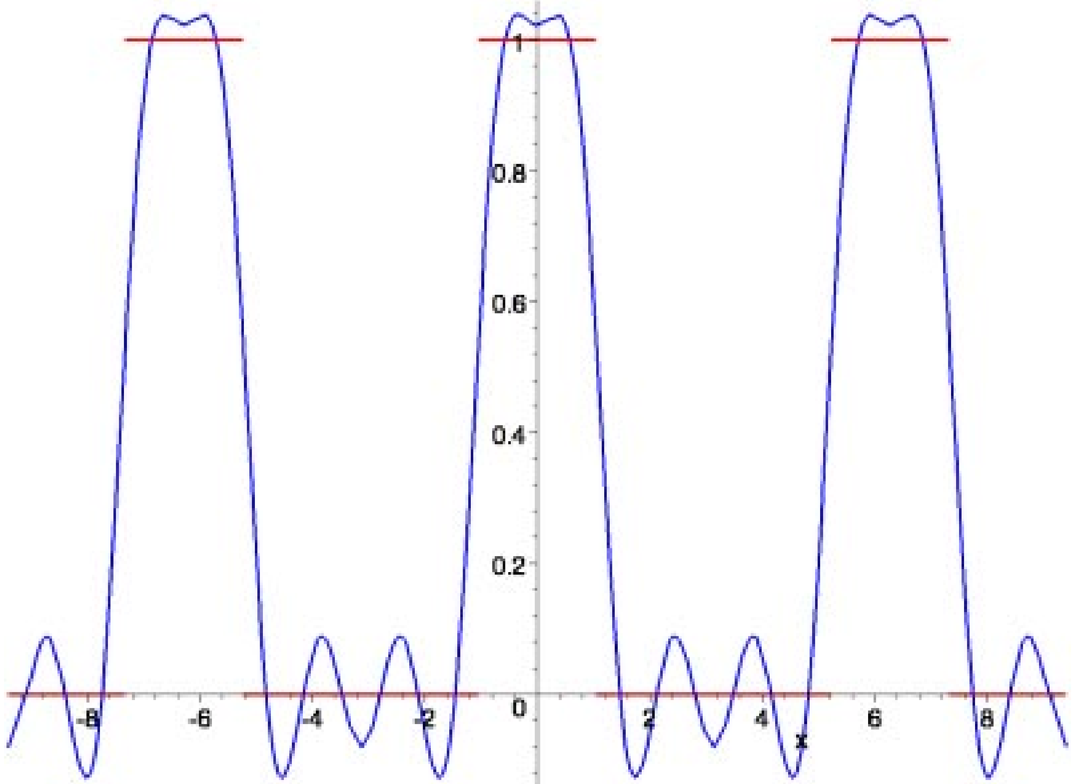

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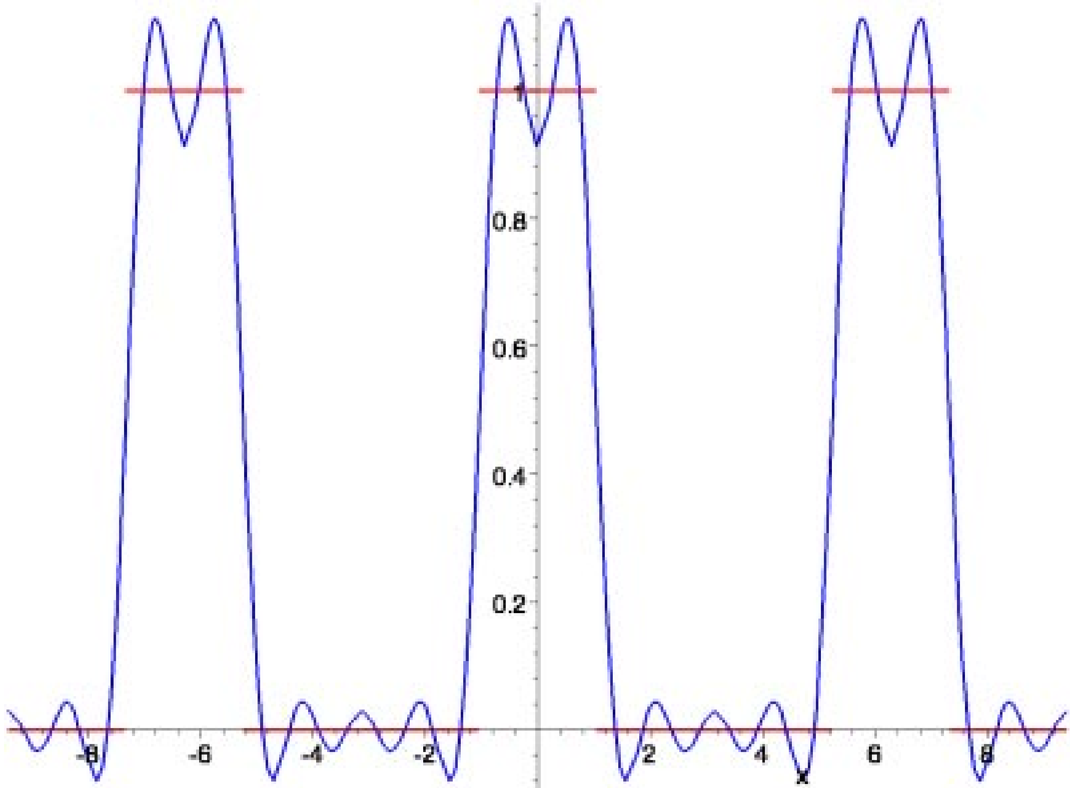

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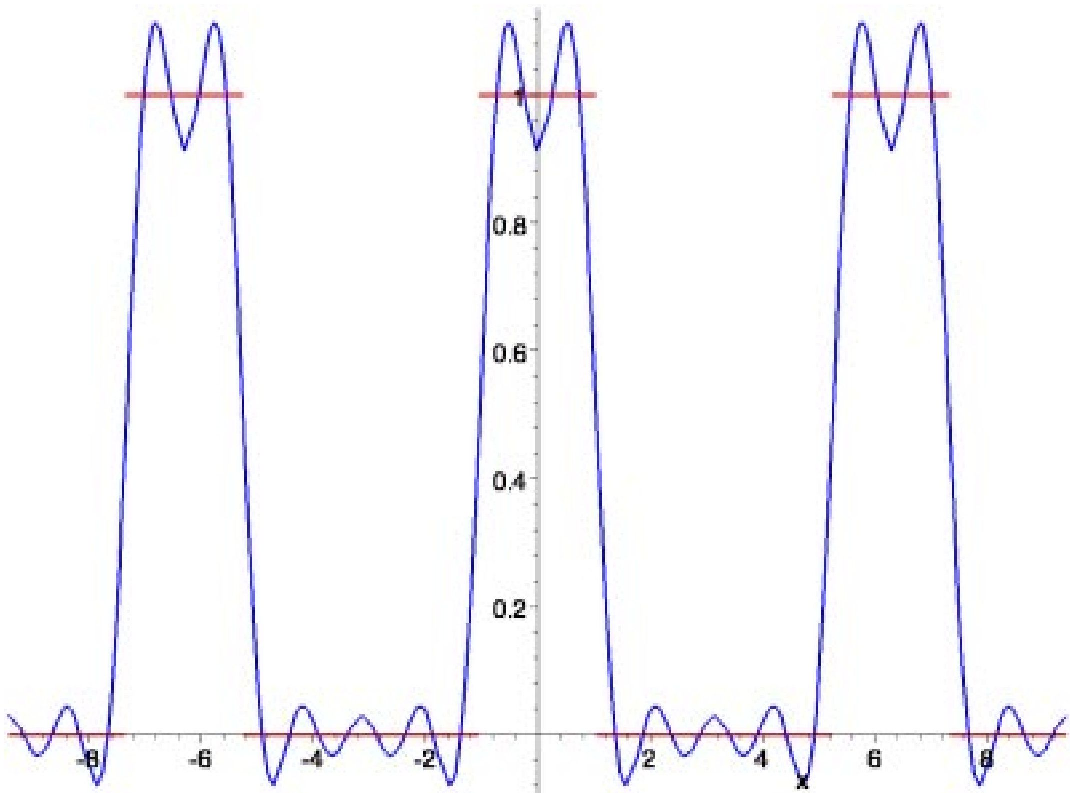

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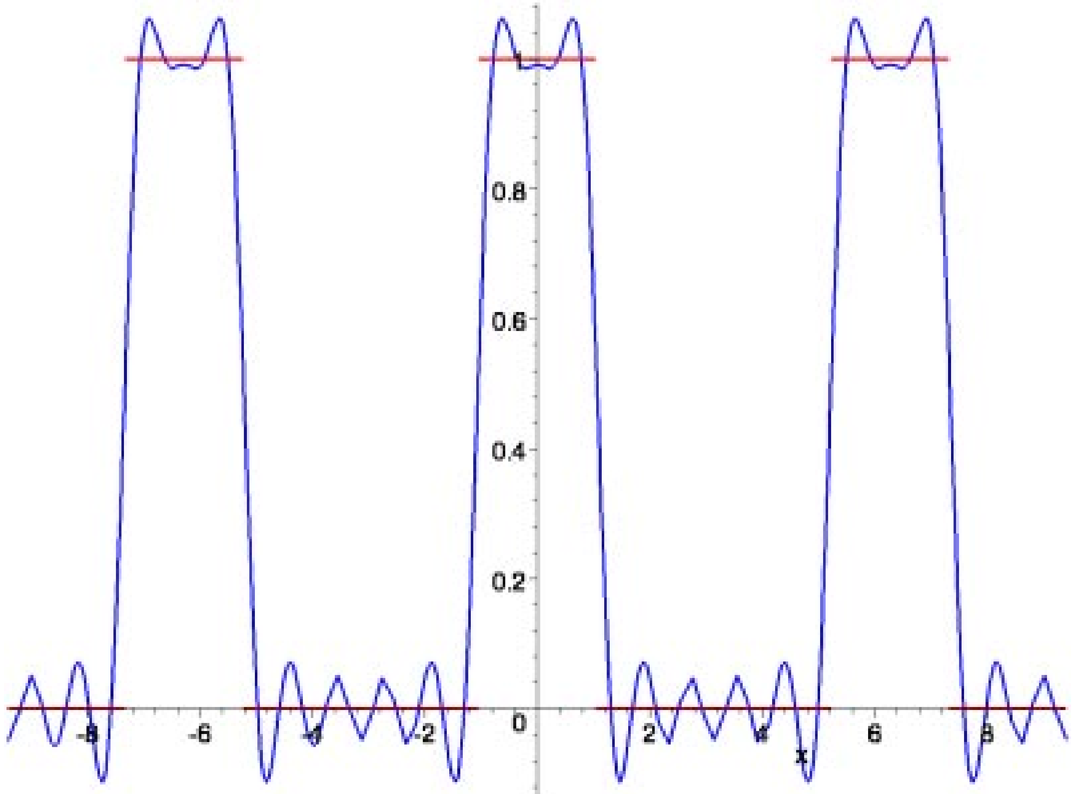

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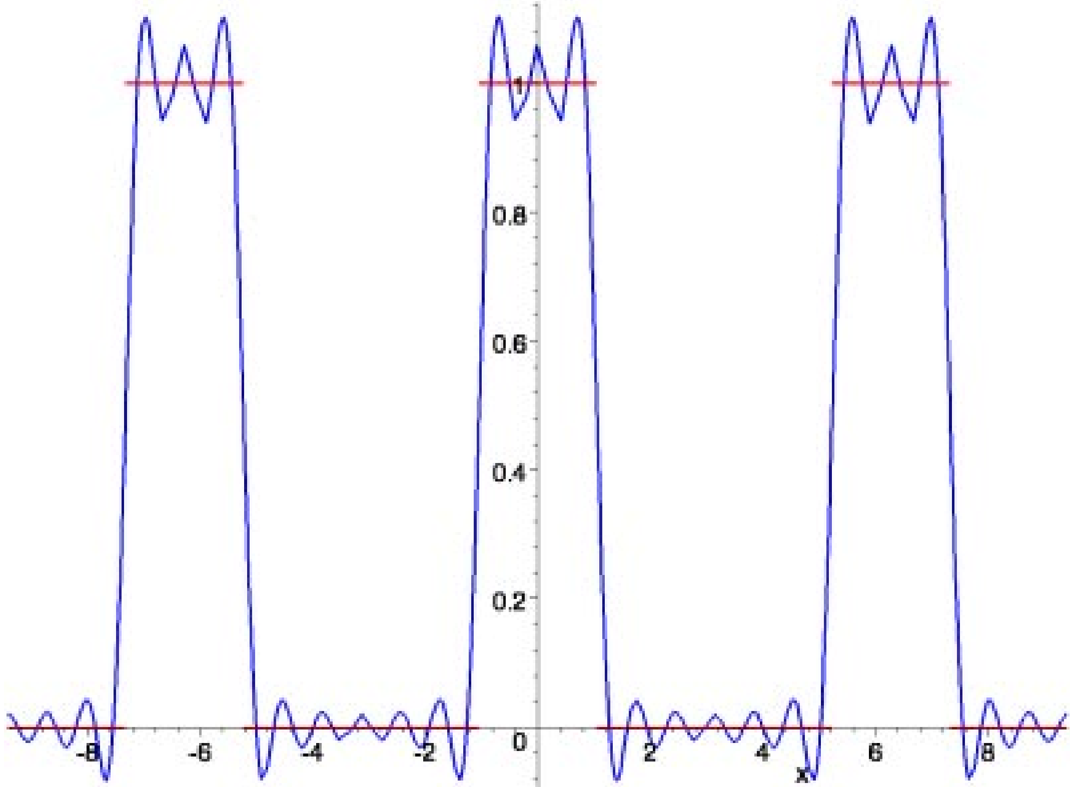

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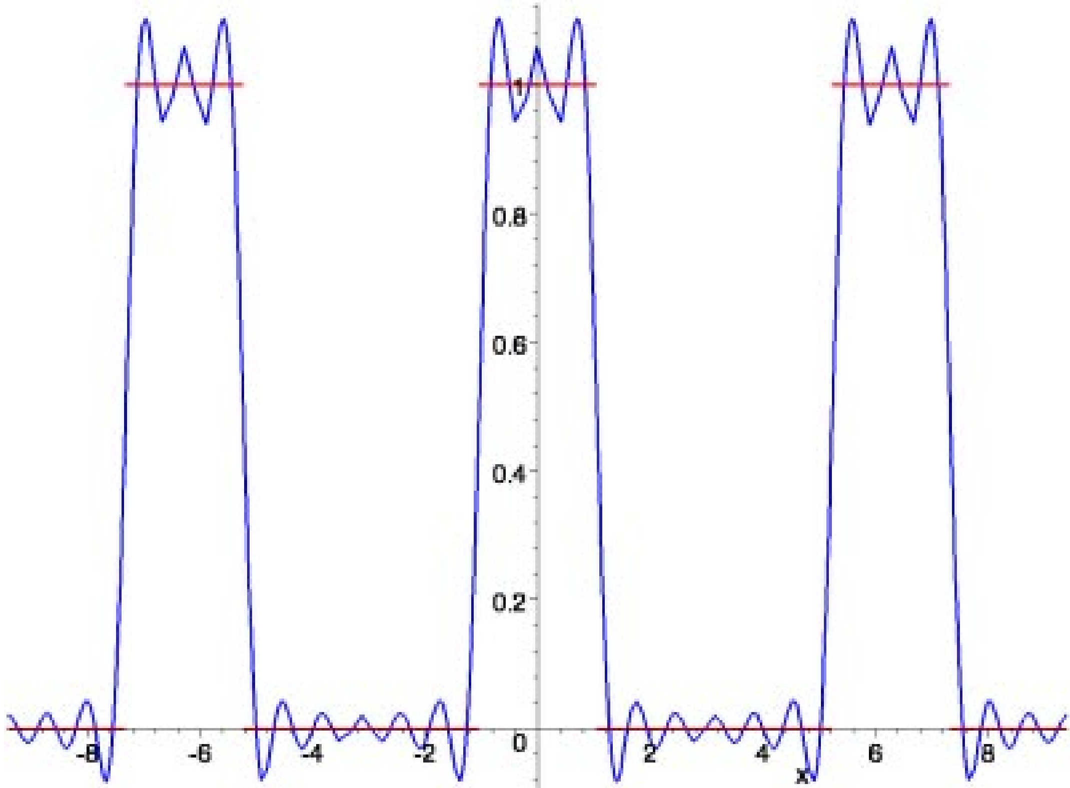

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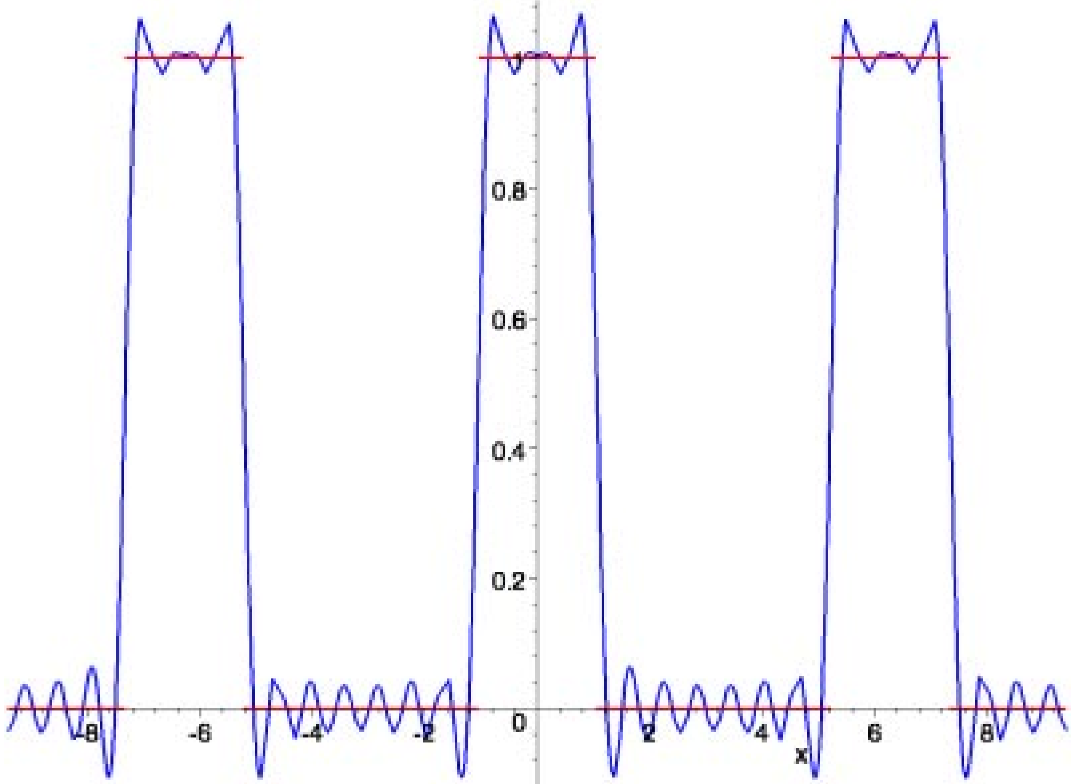

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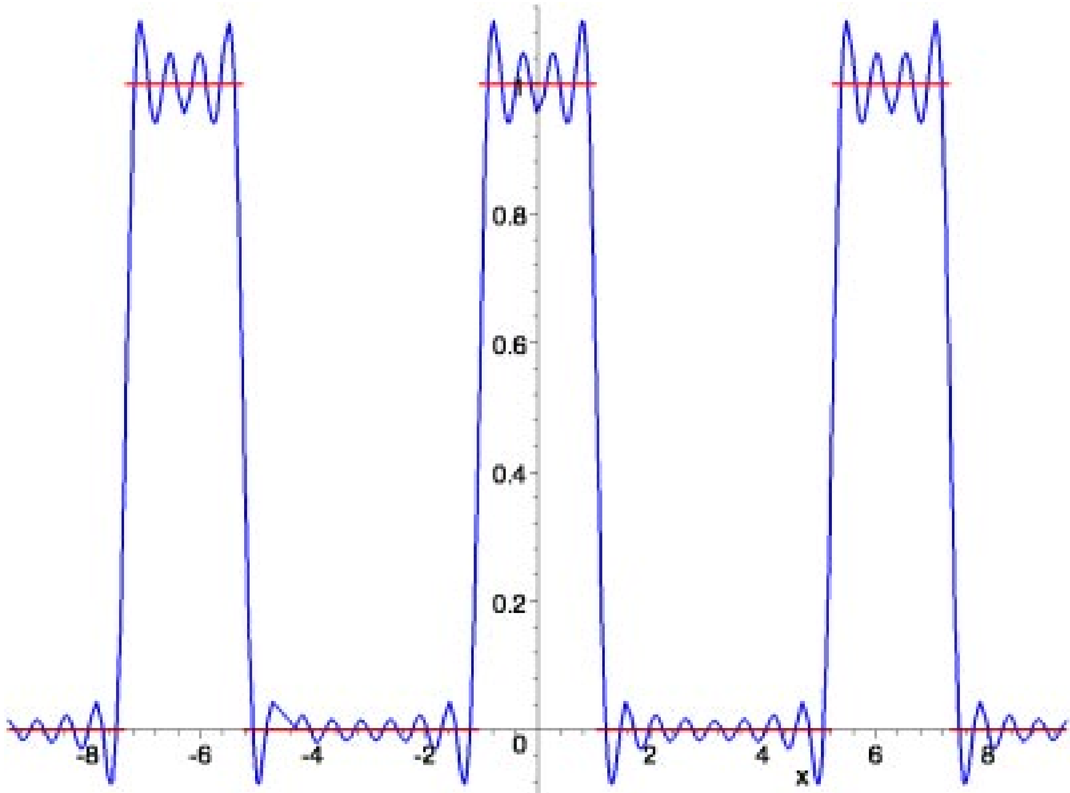

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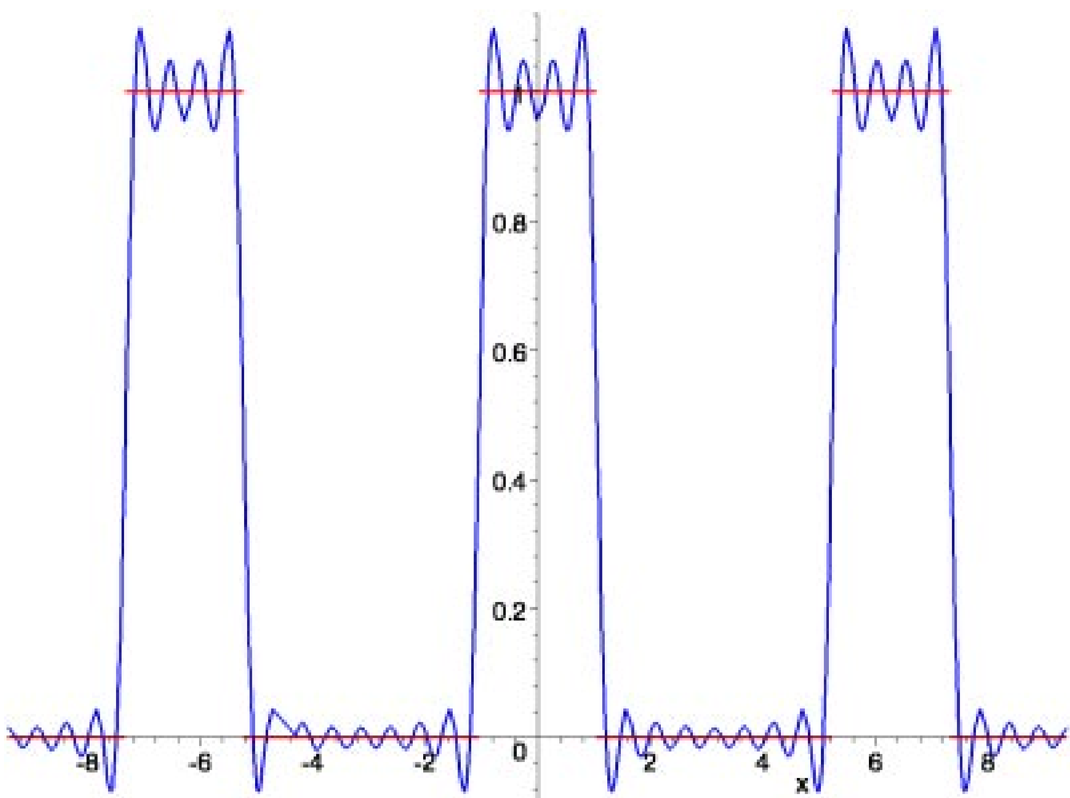

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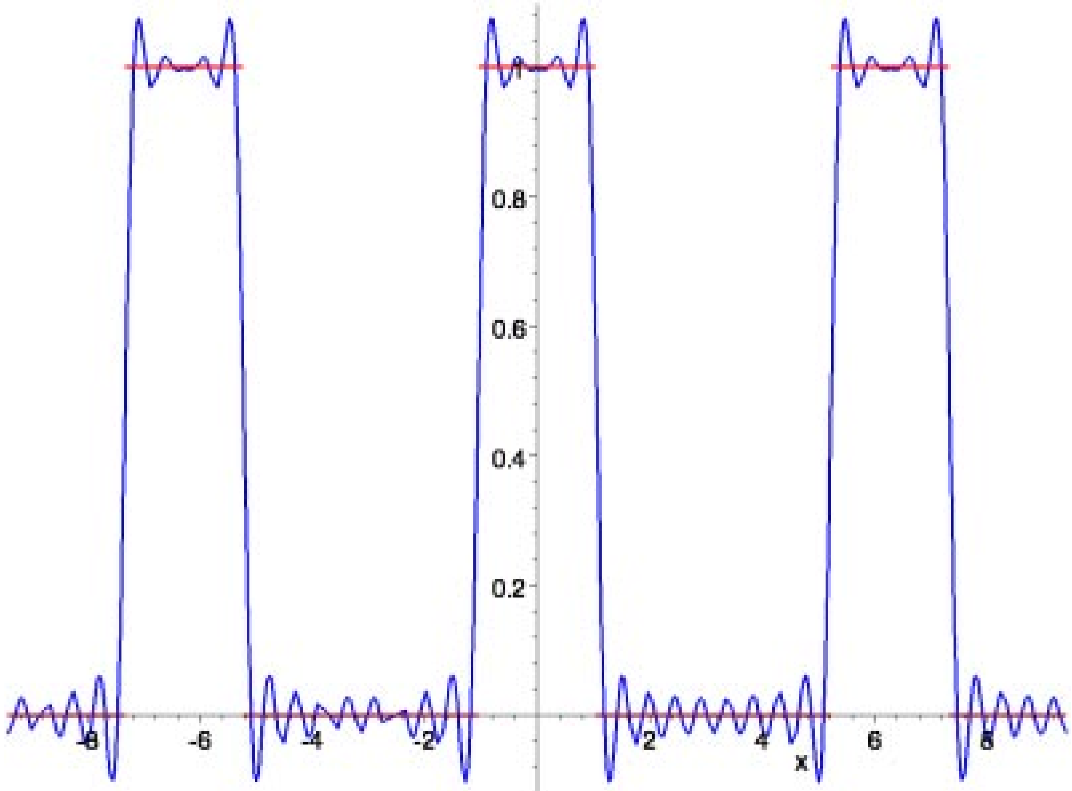
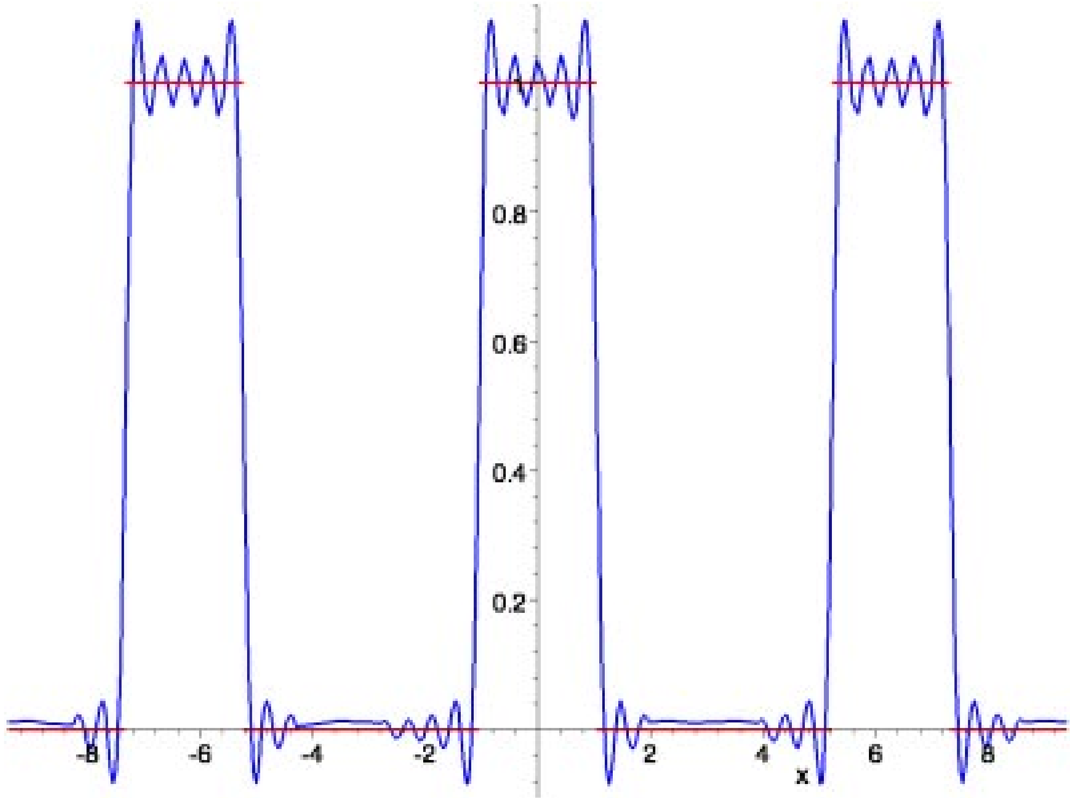
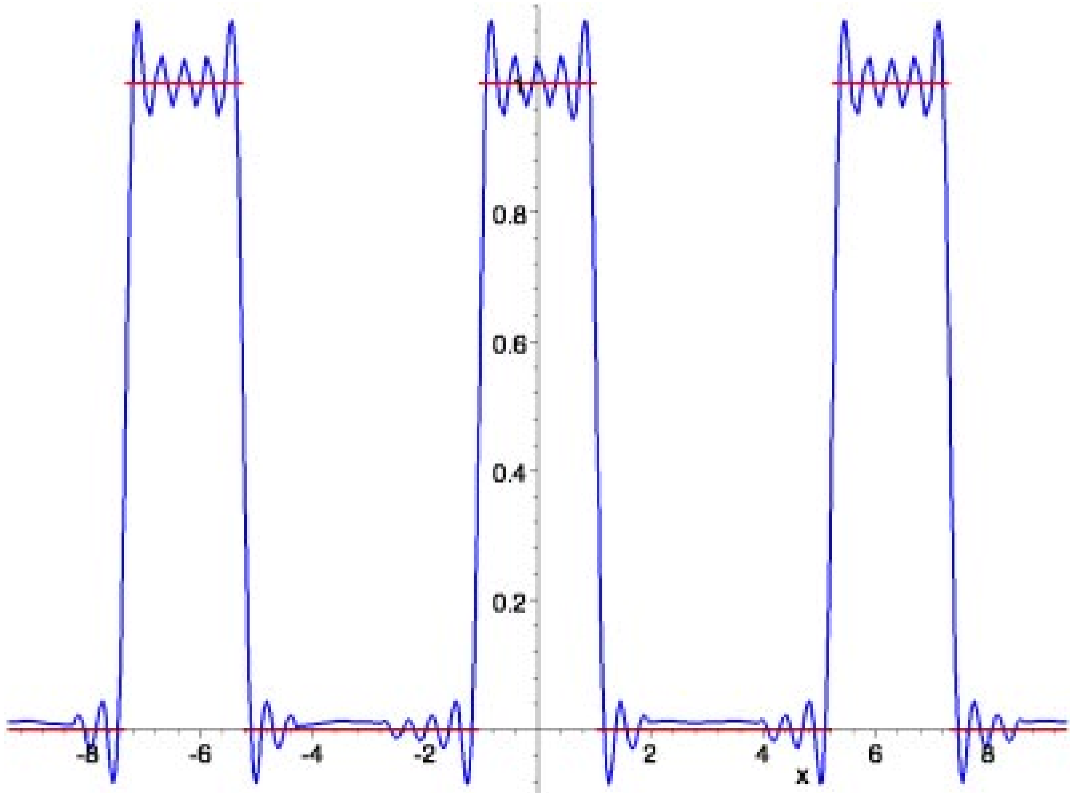


14



15



>

Period *b* over the interval $[-\frac{b}{2}, \frac{b}{2}]$.

>

```
> restart:with(plots):
Warning, the name changecoords has been redefined
```

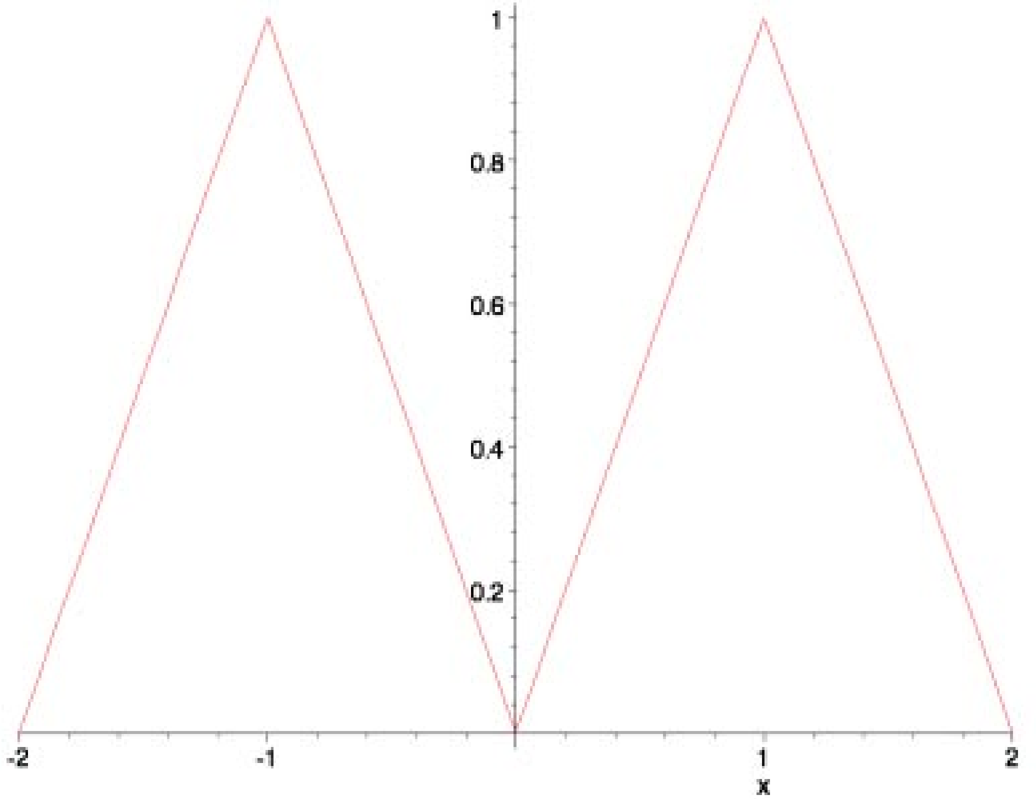
We begin by considering the triangular wave function

```
> f:=piecewise(x<-2,0,x<-1,x+2,x<0,-x,x<1,x,x<2,2-x,0);
```

$$f := \begin{cases} 0 & x < -2 \\ x+2 & x < -1 \\ -x & x < 0 \\ x & x < 1 \\ 2-x & x < 2 \\ 0 & otherwise \end{cases}$$

over the interval x = -2..2 with period *b* = 2, whose plot is below.

```
> p[0]:=plot(f,x=-2..2):
  display(p[0]);
```



We will approximate this function by Fourier polynomials of degrees 1 through 15.

```
> n:=15;
```

$$n := 15$$

We first compute the coefficient $a_0 = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\,dx$.

```
> a[0]:=(1/2)*int(f,x=-1..1);
```

$$a_0 := \frac{1}{2}$$

Next we use a loop to compute the coefficients $a_k = \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\cos\left(\frac{2\pi k x}{b}\right)dx$ and $b_k = \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\sin\left(\frac{2\pi k x}{b}\right)dx$, along with the Fourier polynomials

$F_n(x) = a_0 + \left(\sum_{k=1}^{n} a_k \cos\left(\frac{2\pi k x}{b}\right)\right) + \left(\sum_{k=1}^{n} b_k \sin\left(\frac{2\pi k x}{b}\right)\right)$ for $n = 1..15$, keeping in mind that $b = 2$ here.

```
> for k from 1 to n do
  a[k]:=int(f*cos(Pi*k*x),x=-1..1);
  b[k]:=int(f*sin(Pi*k*x),x=-1..1);
  F[k]:=a[0]+sum('a[i]*cos(Pi*i*x)','i'=1..k)+sum('b[i]*sin(Pi*i*x)','i
  '=1..k);
  od;
```

$$a_1 := -4\frac{1}{\pi^2}$$

$$b_1 := 0$$

$$F_1 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2}$$

$$a_2 := 0$$

$$b_2 := 0$$

$$F_2 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2}$$

$$a_3 := -\frac{4}{9}\frac{1}{\pi^2}$$

$$b_3 := 0$$

$$F_3 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2}$$

$$a_4 := 0$$

$$b_4 := 0$$

$$F_4 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2}$$

$$a_5 := -\frac{4}{25}\frac{1}{\pi^2}$$

$$b_5 := 0$$

$$F_5 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2}$$

$$a_6 := 0$$

$$b_6 := 0$$

$$F_6 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2}$$

$$a_7 := -\frac{4}{49}\frac{1}{\pi^2}$$

$$b_7 := 0$$

$$F_7 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2}$$

$$a_8 := 0$$

$$b_8 := 0$$

$$F_8 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2}$$

$$a_9 := -\frac{4}{81}\frac{1}{\pi^2}$$

$$b_9 := 0$$

$$F_9 := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2}$$

$$a_{10} := 0$$

$$b_{10} := 0$$

$$F_{10} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2}$$

$$a_{11} := -\frac{4}{121}\frac{1}{\pi^2}$$

$$b_{11} := 0$$

$$F_{11} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2} - \frac{4}{121}\frac{\cos(11\pi x)}{\pi^2}$$

$$a_{12} := 0$$

$$b_{12} := 0$$

$$F_{12} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2} - \frac{4}{121}\frac{\cos(11\pi x)}{\pi^2}$$

$$a_{13} := -\frac{4}{169}\frac{1}{\pi^2}$$

$$b_{13} := 0$$

$$F_{13} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2} - \frac{4}{121}\frac{\cos(11\pi x)}{\pi^2} - \frac{4}{169}\frac{\cos(13\pi x)}{\pi^2}$$

$$a_{14} := 0$$

$$b_{14} := 0$$

$$F_{14} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2} - \frac{4}{121}\frac{\cos(11\pi x)}{\pi^2} - \frac{4}{169}\frac{\cos(13\pi x)}{\pi^2}$$

$$a_{15} := -\frac{4}{225}\frac{1}{\pi^2}$$

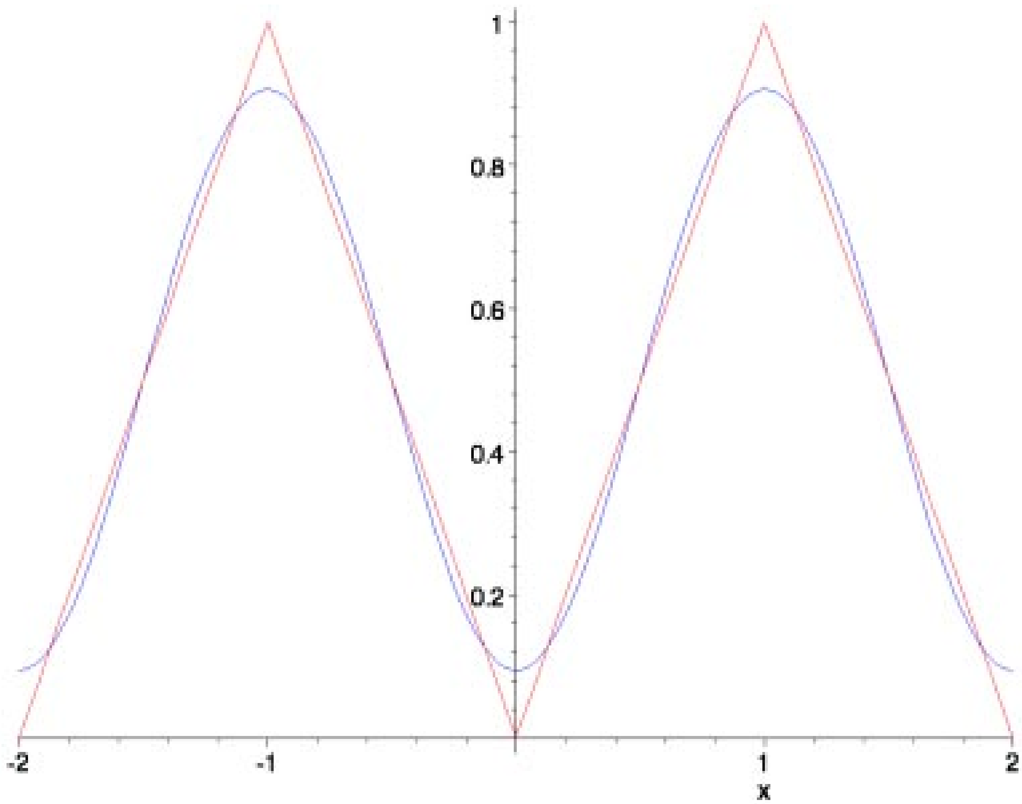
$$b_{15} := 0$$

$$F_{15} := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} - \frac{4}{49}\frac{\cos(7\pi x)}{\pi^2} - \frac{4}{81}\frac{\cos(9\pi x)}{\pi^2} - \frac{4}{121}\frac{\cos(11\pi x)}{\pi^2} - \frac{4}{169}\frac{\cos(13\pi x)}{\pi^2} - \frac{4}{225}\frac{\cos(15\pi x)}{\pi^2}$$

Next we plot f(x) along with each of the $F_n(x)$ for n = 1..15.

```
> for k from 1 to n do
  p[k]:=plot(F[k],x=-2..2,color=blue):
  od:
> for k from 1 to n do
  k;
  display(p[0],p[k]):
  od;
```

1



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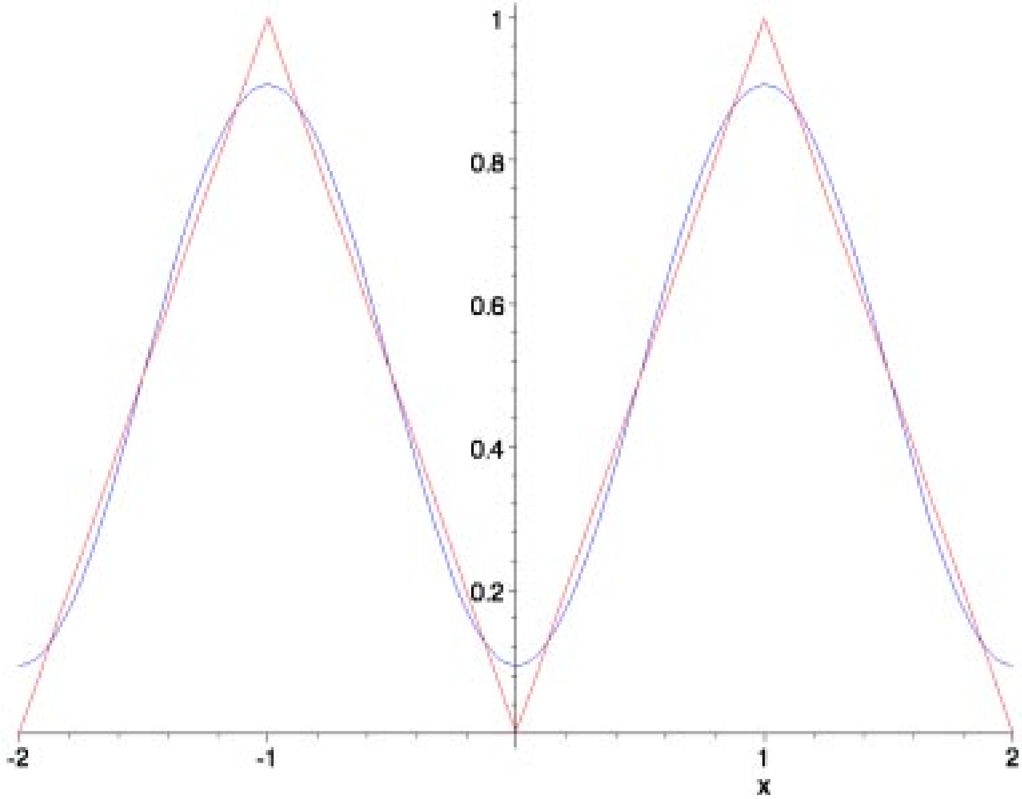

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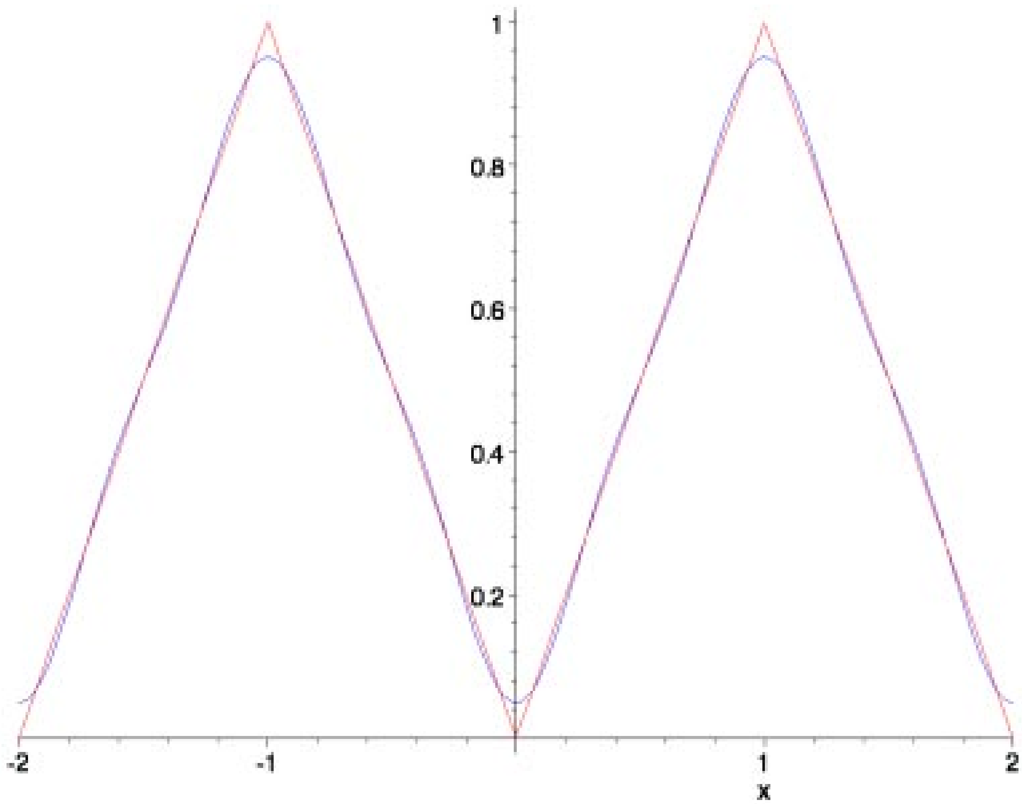

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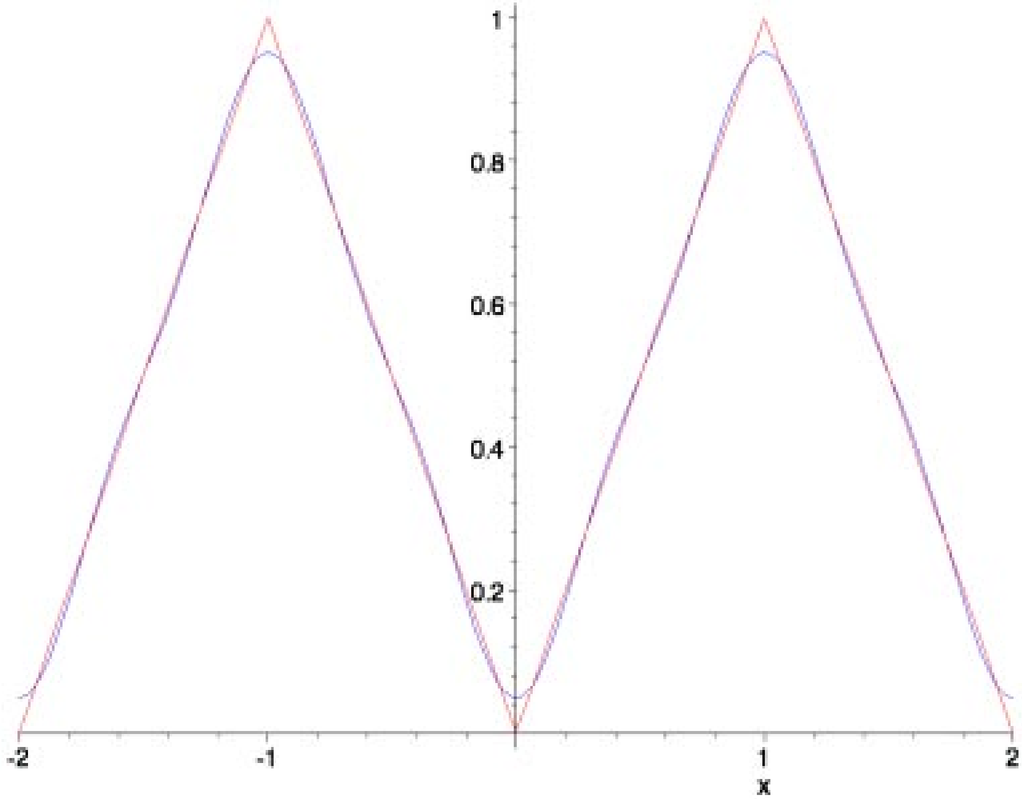

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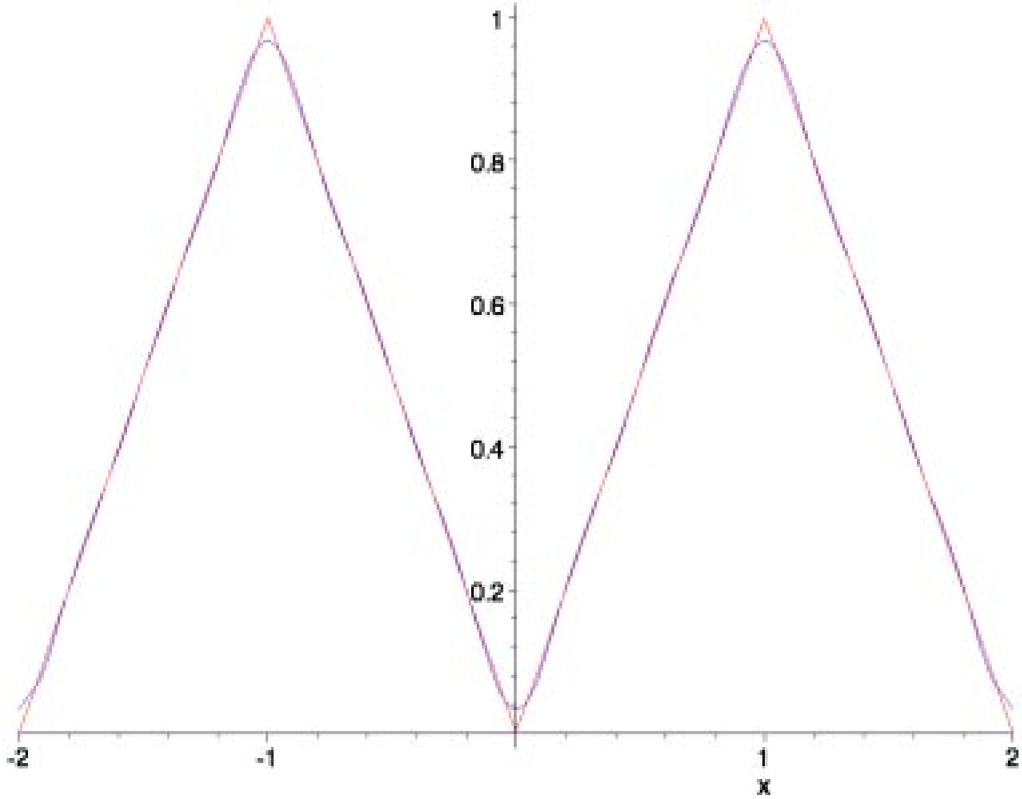

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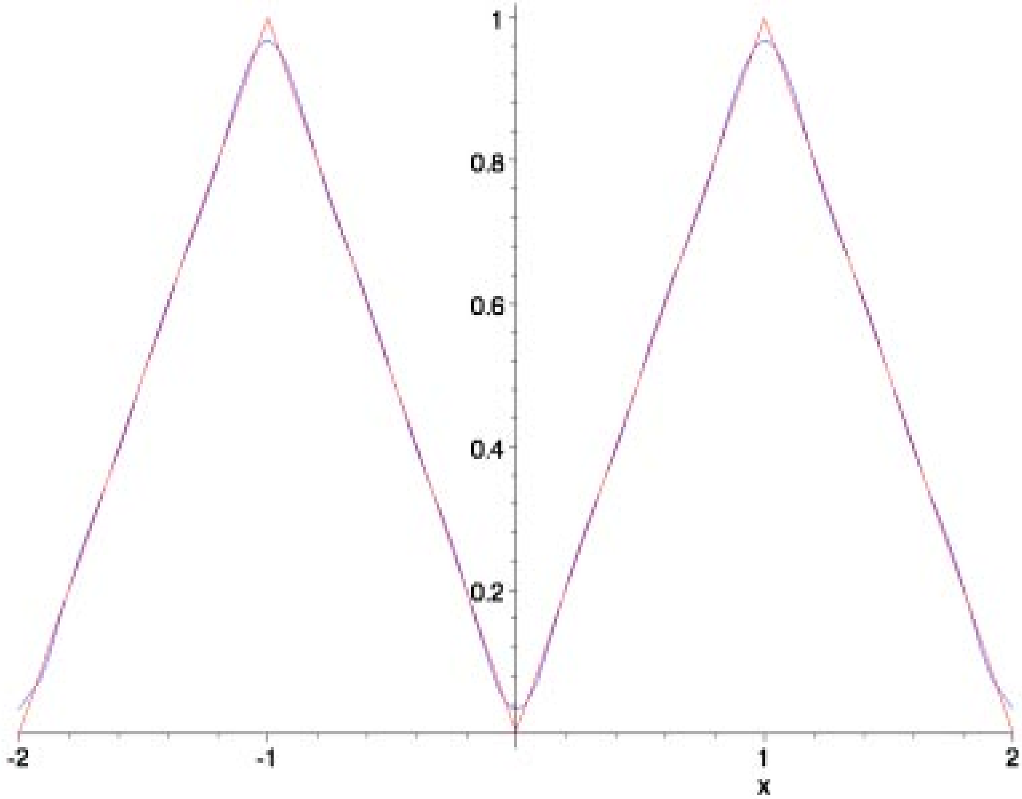

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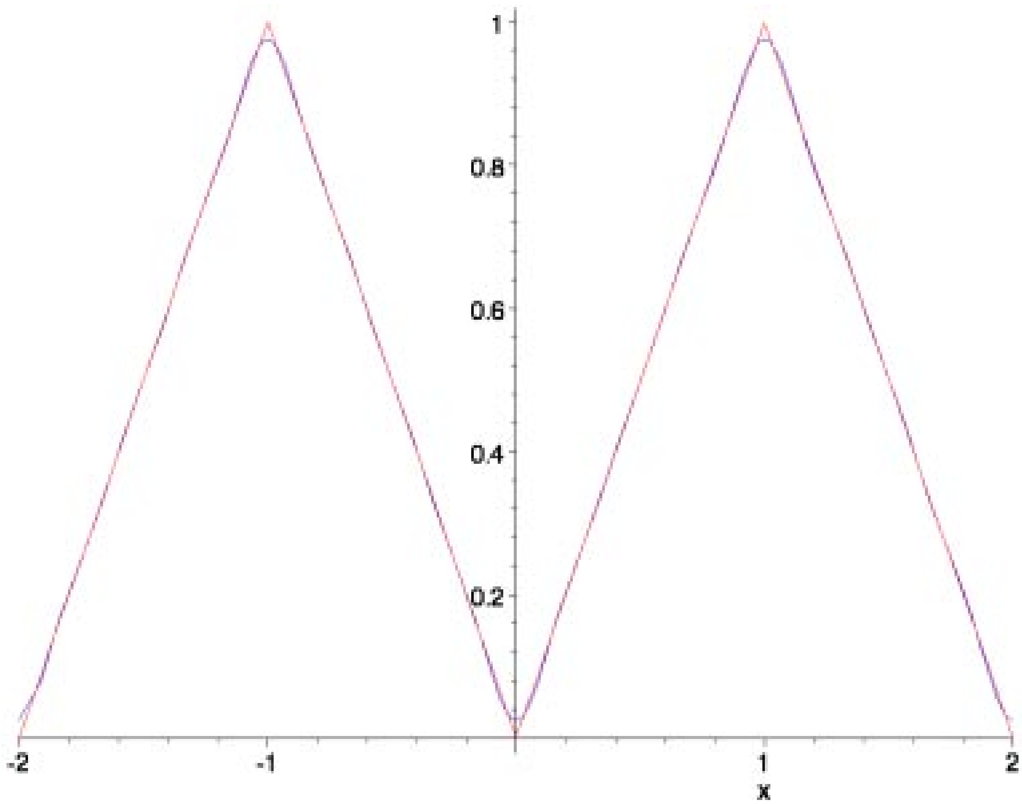

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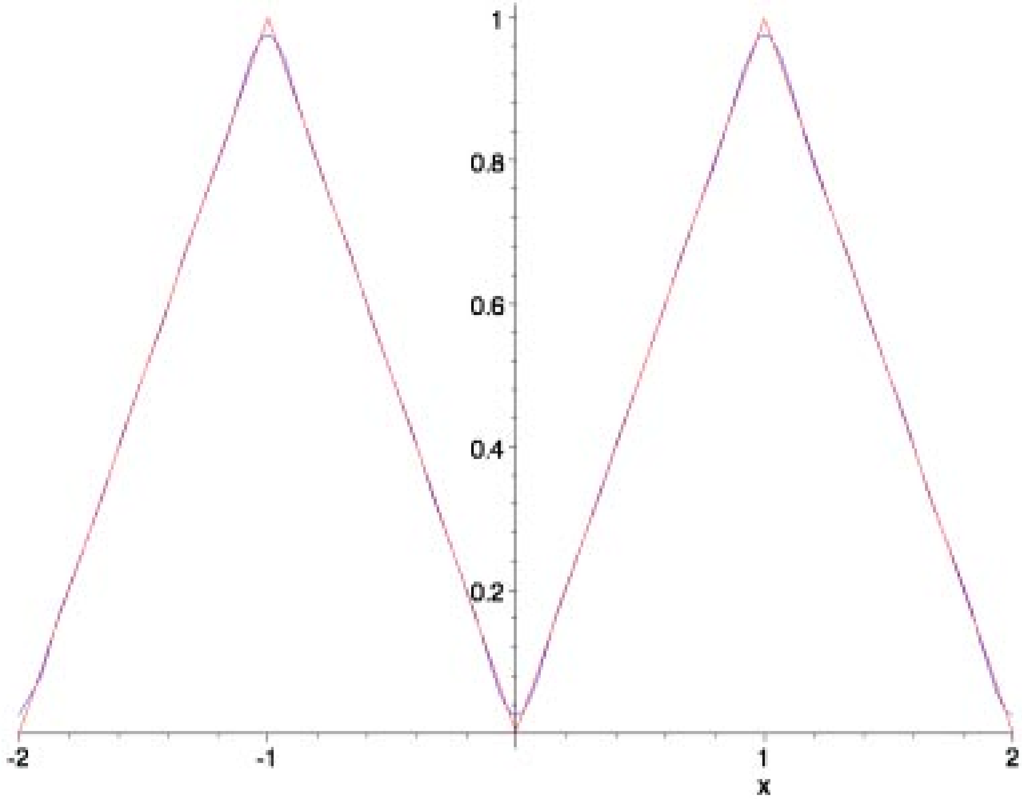

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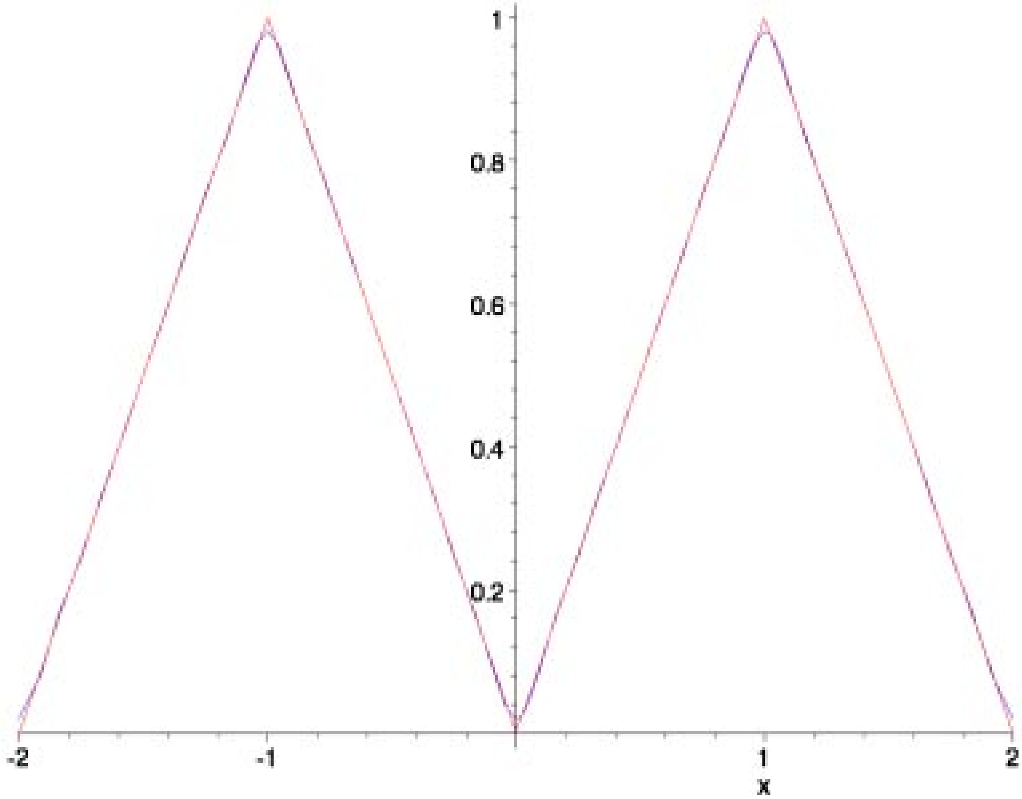

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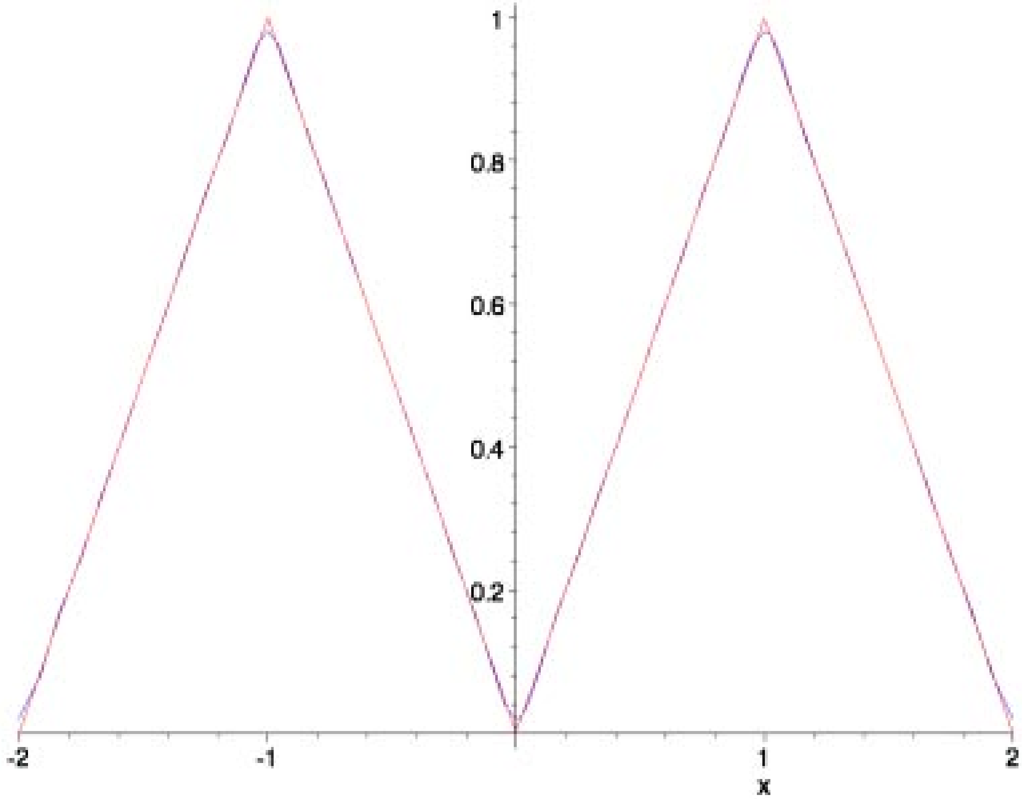

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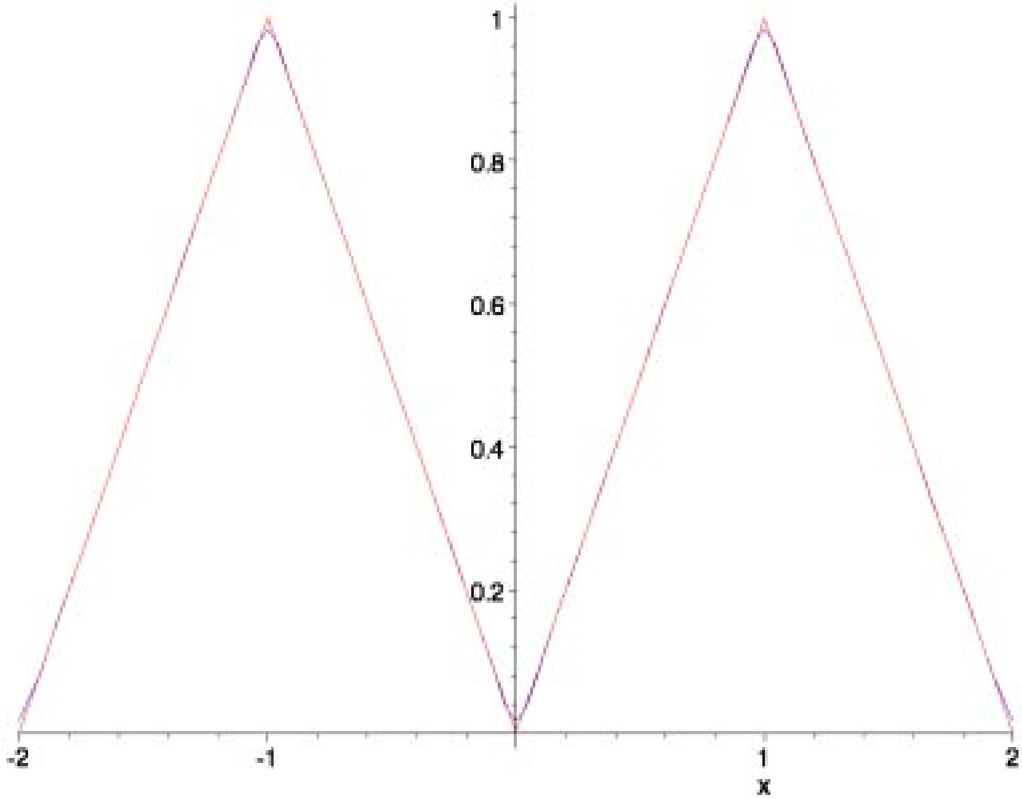

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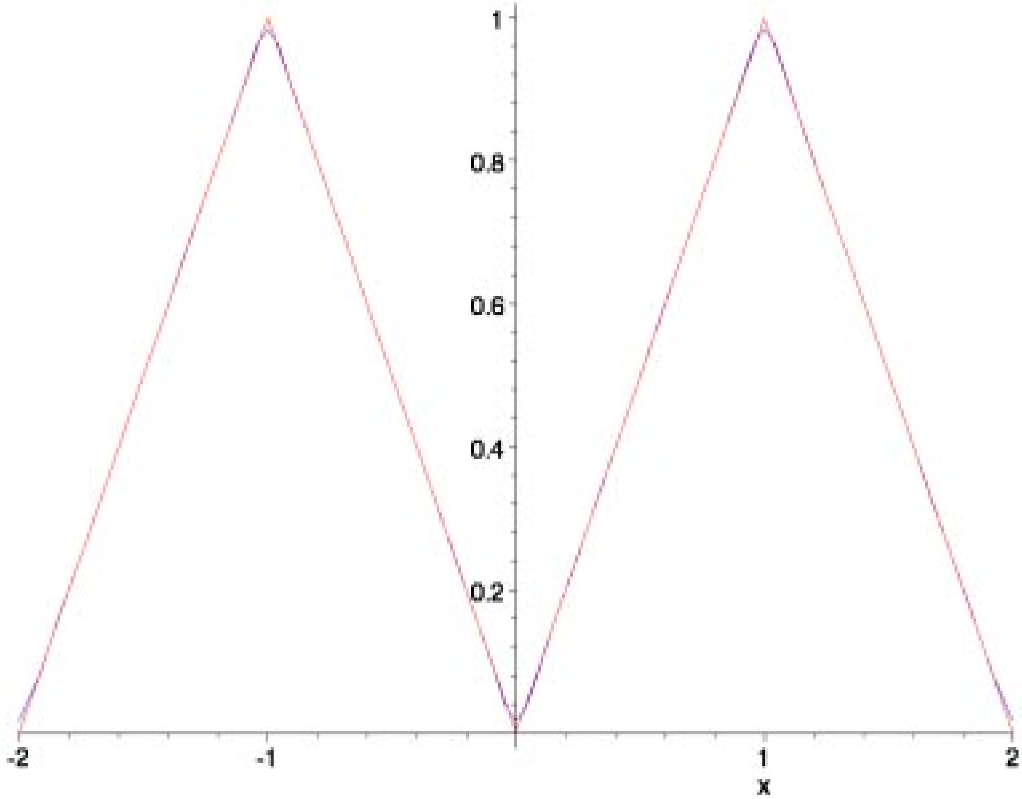

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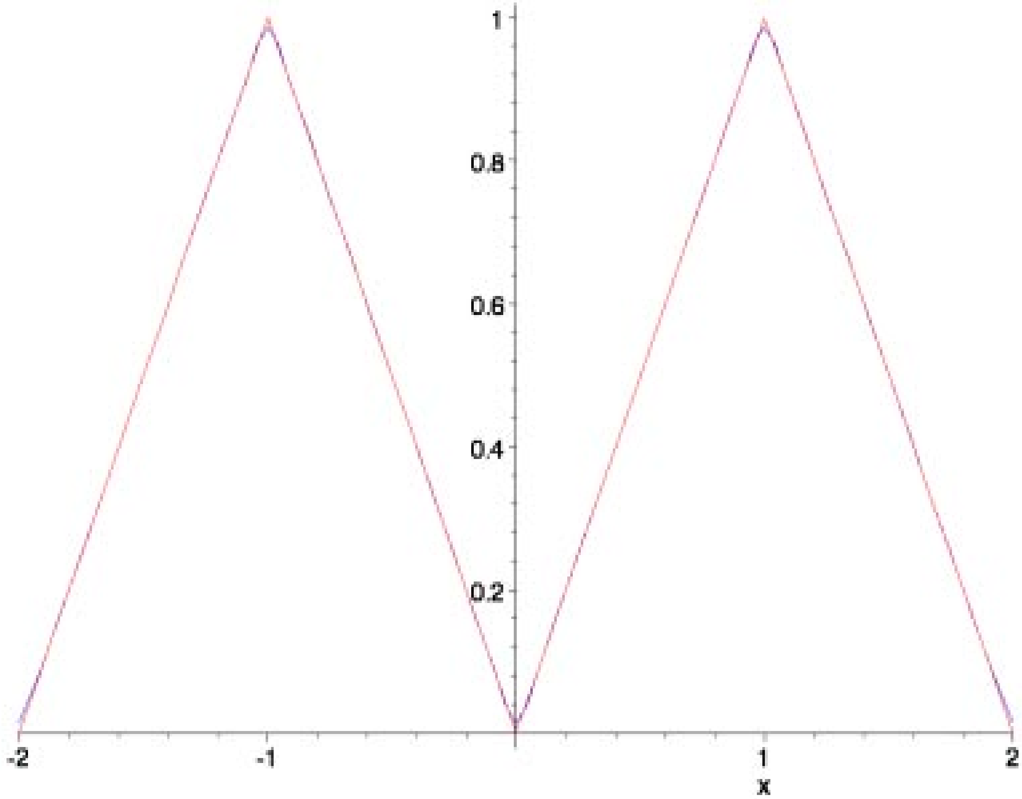

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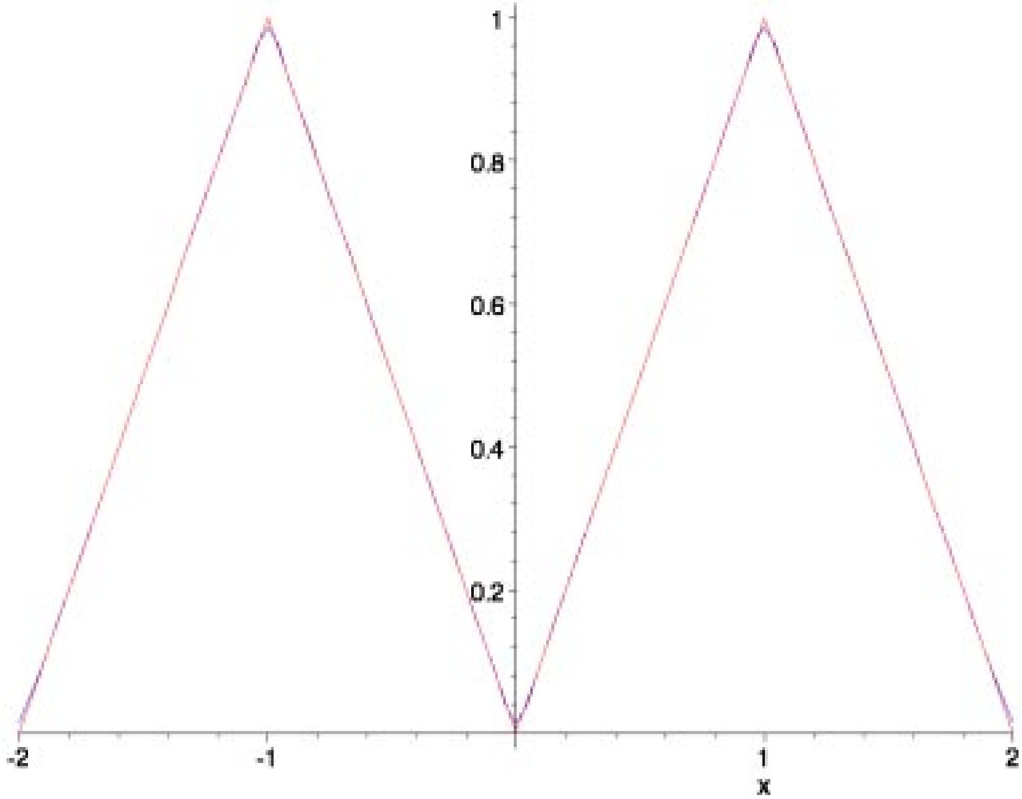

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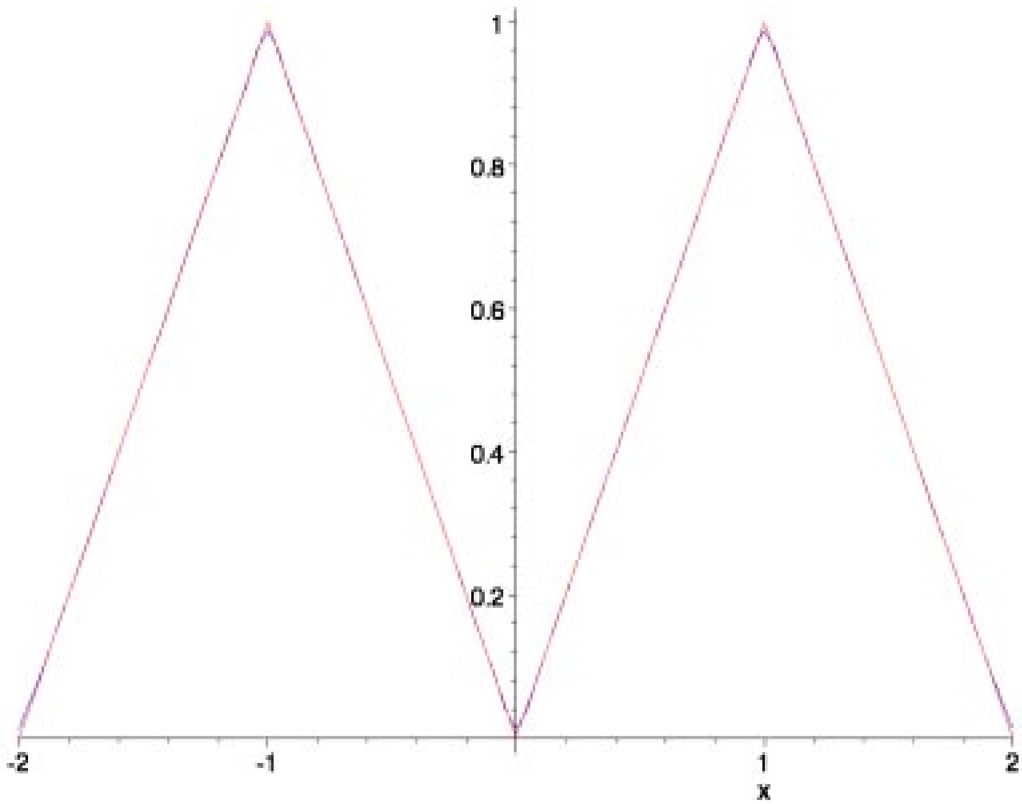

Again, we see that the successive Fourier polynomials give better and better approximations to f(x).

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