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Taylor Polynomials

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Taylor Polynomials about x = 0.

A primary use of Taylor polynomials is to find good polynomial approximations to a function near a specified value. As a first example, we use a fourth degree Taylor polynomial to approximate $f(x) = \mathbf{e}^x$ near $x = 0$. We begin by entering our function as a Maple expression.

```
> restart;with(plots);
Warning, the name changecoords has been redefined
```

[*animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot*]

```
> f:=exp(x);
```

$$f := \mathbf{e}^x$$

We first use the comand taylor to form the Taylor series, which we will discuss later, for the function. The first parameter is the function expression, the second, assuming x is our variable, is "x =" the number we want to approximate near, and the third is an integer one higher than the degree of the Taylor polynomial we desire. We desire degree 4 here, so use 5.

```
> T4:=taylor(f,x=0,5);
```

$$T4 := 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \mathrm{O}(x^5)$$

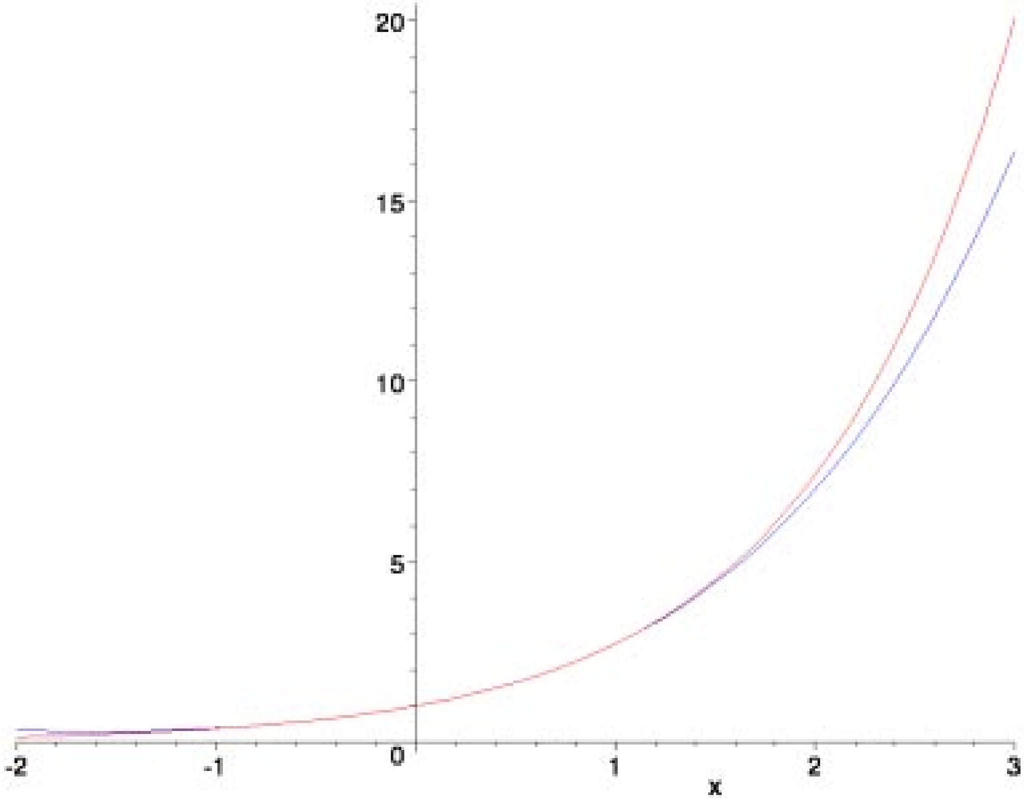
This is the Taylor polynomial of degree 4 with a remainder term $\mathrm{O}(x^5)$. This remainder term is pronounced as "big Oh of x^5." To get the Taylor polynomial as an expression from the Taylor series, we must convert the series *T4* to a polynomial using convert.

```
> P4:=convert(T4,polynom);
```

$$P4 := 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

Now let's plot both the function (red) with the Taylor polynomial (blue).

```
> plot0:=plot(f,x=-2..3,color=red):
> plot1:=plot(P4,x=-2..3,color=blue):
> display(plot0,plot1);
```



We see that the graphs are basically the same for *x* between -1 and 1, but start to differ to the left and right of this interval. Let's check a couple of values for *x*. We work out from 0, checking the difference at .5, 1, and 1.5.

```
> evalf(subs(x=.5,f-P4));
```

.000283771

```
> evalf(subs(x=1,f-P4));
```

.009948495

```
> evalf(subs(x=1.5,f-P4));
```

.083251570

Note that the approximations are quite good near 0, but lose accuracy as one moves away from 0. We see the same effect as we move to the left.

```
> evalf(subs(x=-.5,f-P4));
```

-.0002401737

```
> evalf(subs(x=-1,f-P4));
```

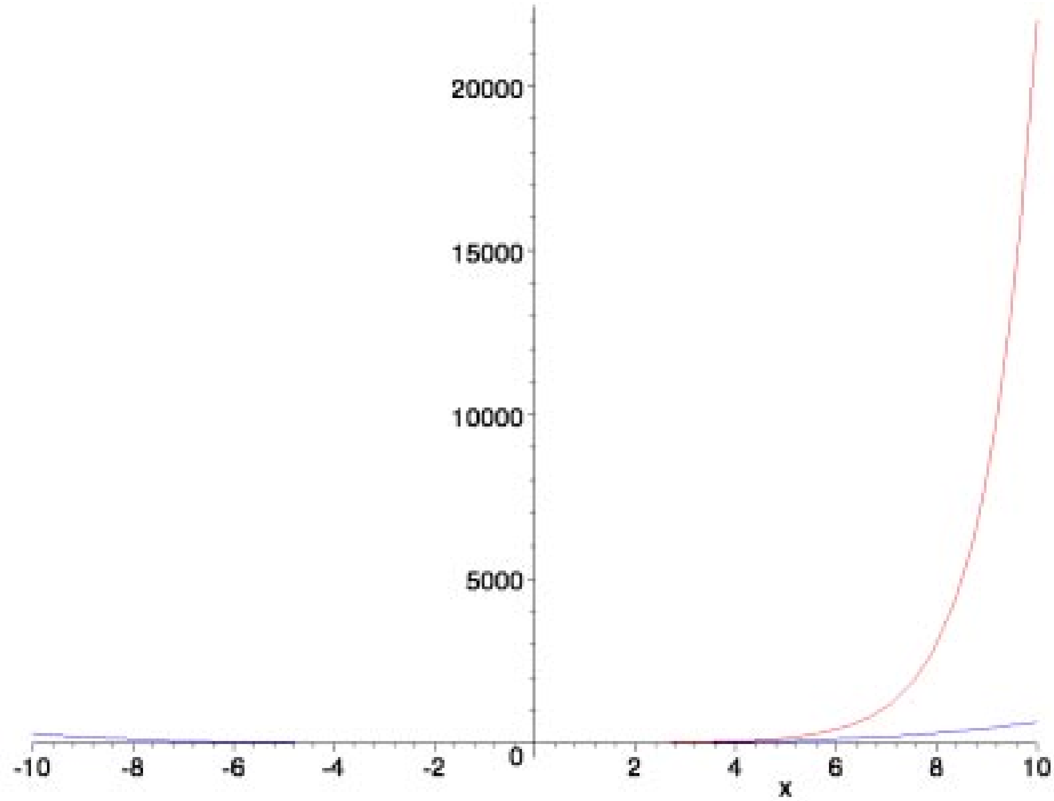
-.0071205588

```
> evalf(subs(x=-1.5,f-P4));
```

-.0503073399

Now let's look at a larger window.

```
> plot0:=plot(f,x=-10..10,color=red):
> plot1:=plot(P4,x=-10..10,color=blue):
> display(plot0,plot1);
```



We now see considerable difference between the two graphs away from 0, In fact they differ by more than 20,000 at $x = 10$.

>

Suppose f has n derivatives at 0. Then the Taylor polynomial of degree n about $x = 0$ for f is

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^k(0)\, x^k.$$

This polynomial has the same value as the function at 0, and it's first n derivatives have the same value at 0 as do the corresponding derivatives of the function. Let's examine Taylor polynomials further by using the function $f(x) = \ln(x + 1)$.

```
> f:=ln(x+1);
```

$$f := \ln(x + 1)$$

For this (and any) function, the Taylor polynomial of degree 0 is the constant polynomial that has the same value as f at 0.

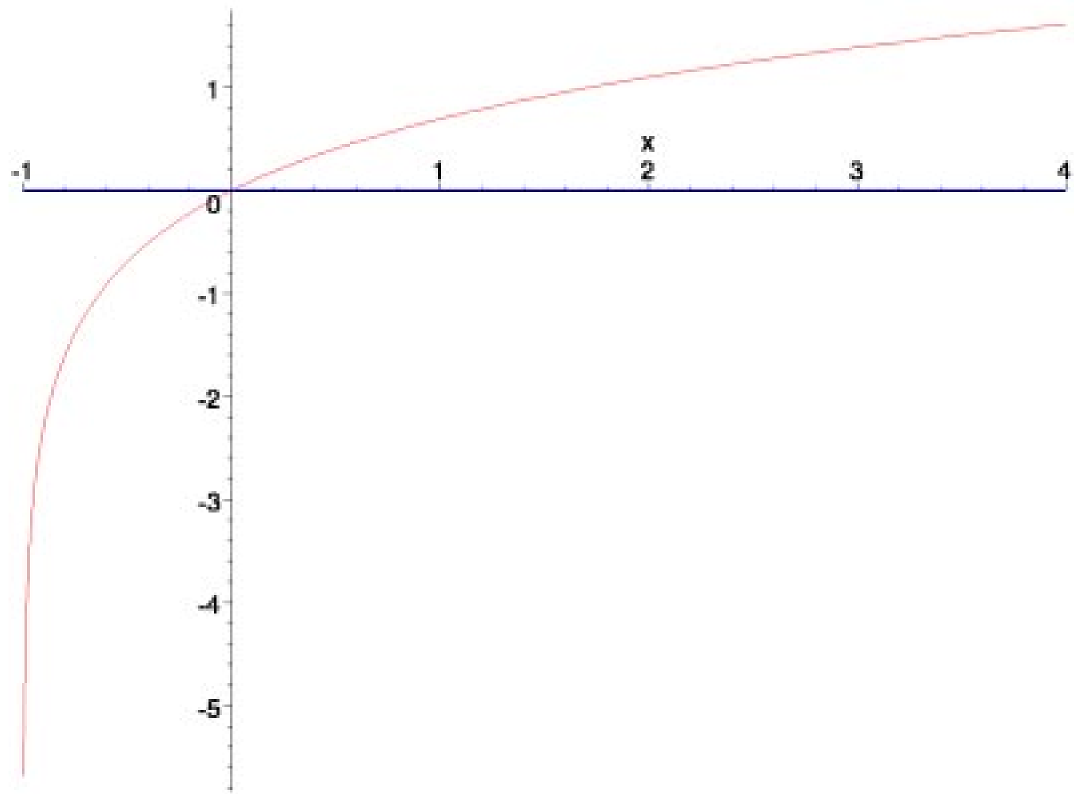
```
> T0:=taylor(f,x = 0,1);
> p0 := convert(T0,polynom);
```

$$T0 := O(x)$$

$$p0 := 0$$

We plot the function (in red) along with the Taylor polynomial (in blue).

```
> plot0:=plot(f,x=-1..4,color=red):
> plot1:=plot(p0,x=-1..4,color=blue,thickness=3):
> display(plot0,plot1);
```



Note that the approximation is not very good except very near $x = 1$. Let's see what happens as we move up a degree to a first order Taylor polynomial.

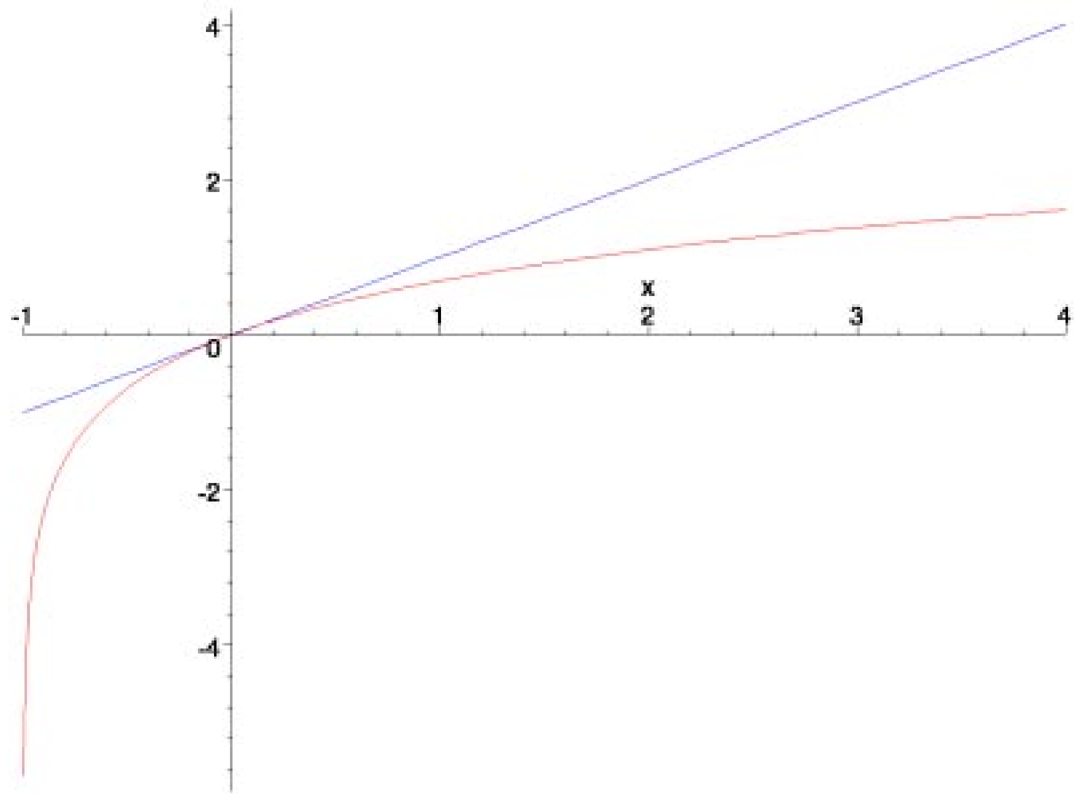
```
> T1:=taylor(f,x = 0,2);
> p1 := convert(T1,polynom);
```

$$T1 := x + O(x^2)$$

$$p1 := x$$

This is also the linear approximation to f at $x = 0$. In the plot below, we see that this linear approximation, which also has the same derivative as f at $x = 0$, is better than the constant approximation.

```
> plot0:=plot(f,x=-1..4,color=red):
> plot1:=plot(p1,x=-1..4,color=blue):
> display(plot0,plot1);
```



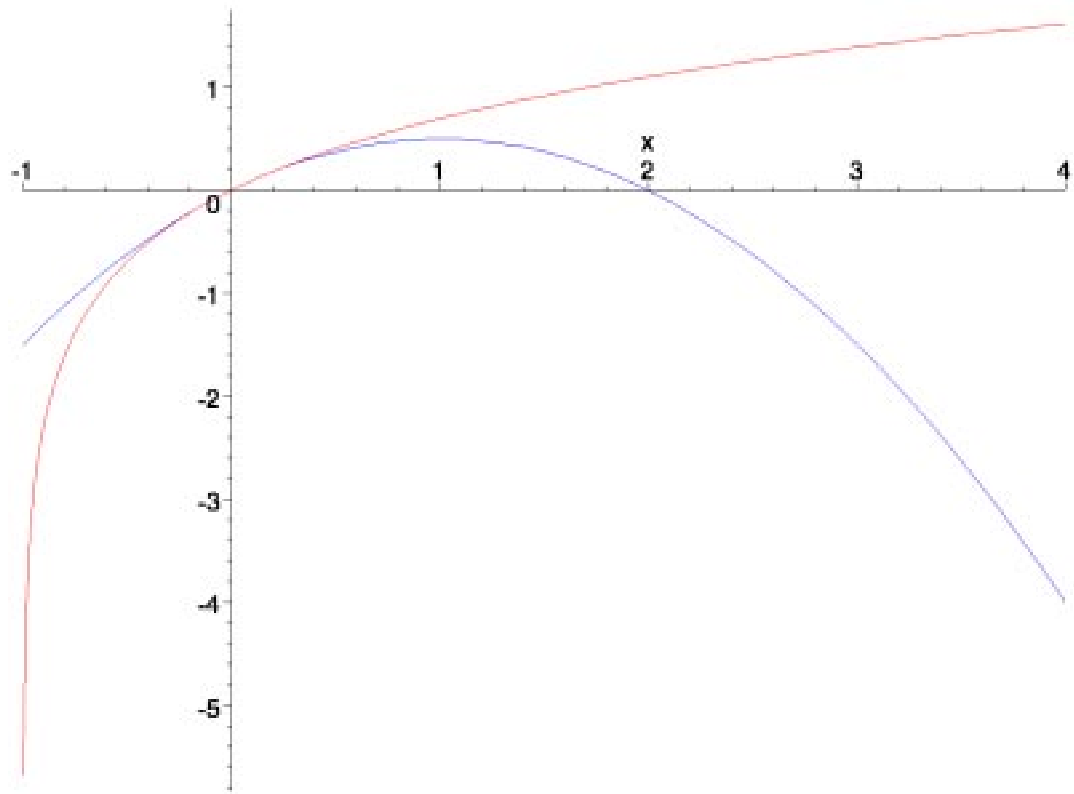
When we advance to the second degree Taylor Polynomial, where the function and the polynomial have the same second derivative at 0, our quadratic approximation gets even better since the polynomial has the same concavity as the function near 0.

```
> T2:= taylor(f,x = 0,3);
> p2:=simplify(convert(T2,polynom));
```

$$T2 := x - \frac{1}{2}x^2 + O(x^3)$$

$$p2 := x - \frac{1}{2}x^2$$

```
> plot0:=plot(f,x=-1..4,color=red):
> plot1:=plot(p2,x=-1..4,color=blue):
> display(plot0,plot1);
```



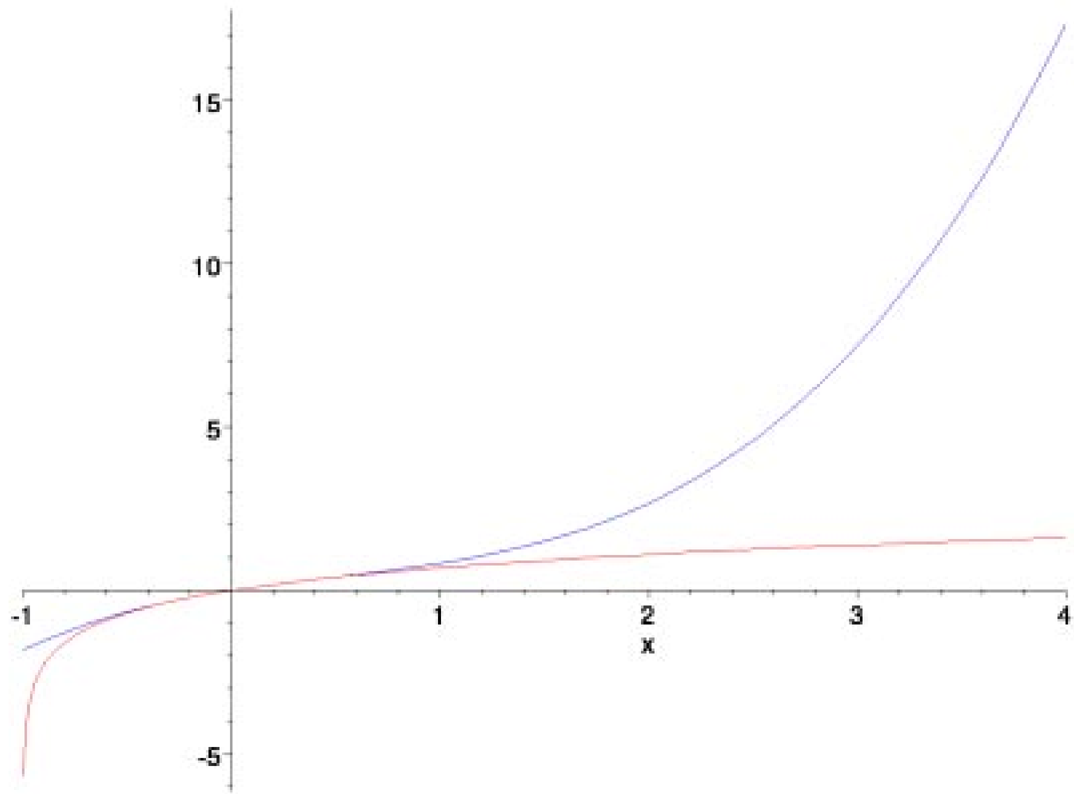
Notice that we have very good agreement from -.2 to .6. Let's go the the third degree Taylor polynomial.

```
> T3:= taylor(f,x = 0,4);
> p3:=simplify(convert(T3,polynom));
```

$$T3 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

$$p3 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

```
> plot0:=plot(f,x=-1..4,color=red):
> plot1:=plot(p3,x=-1..4,color=blue):
> display(plot0,plot1);
```



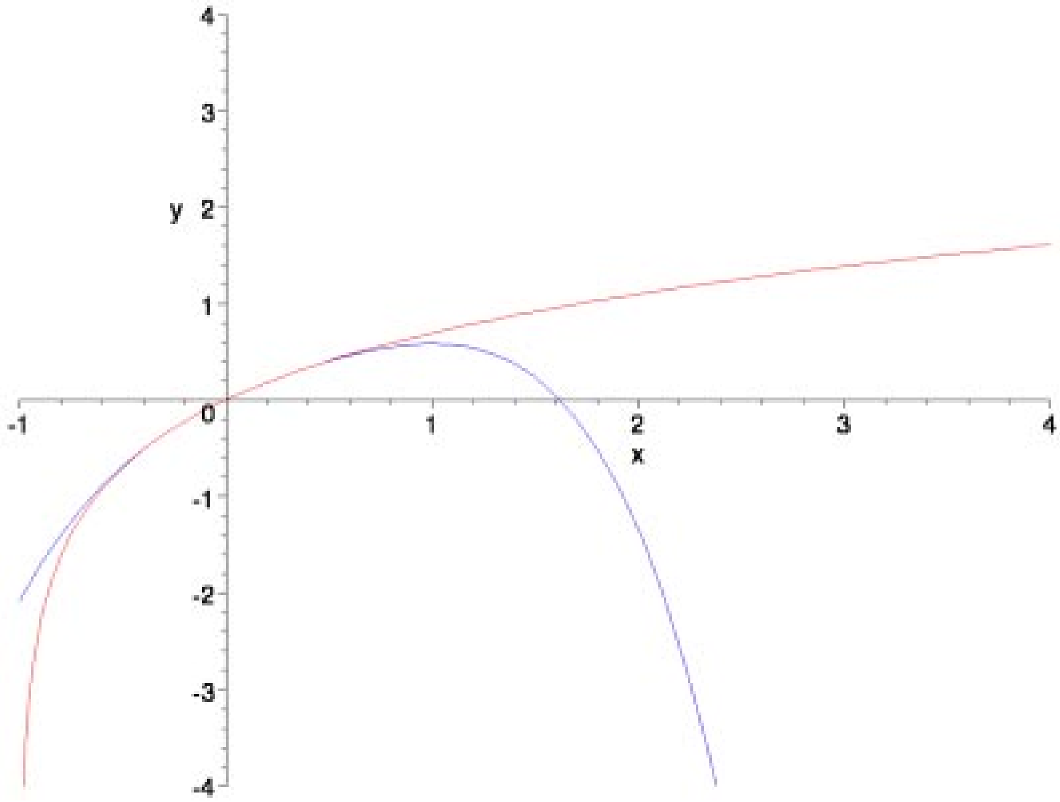
Now we have good agreement from -.6 to .8. We check degree 4.

```
> T4:= taylor(f,x = 0,5);
> p4:=simplify(convert(T4,polynom));
```

$$T4 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

$$p4 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

```
> plot0:=plot(f,x=-1..4,y=-4..4,color=red):
> plot1:=plot(p4,x=-1..4,y=-4..4,color=blue):
> display(plot0,plot1);
```



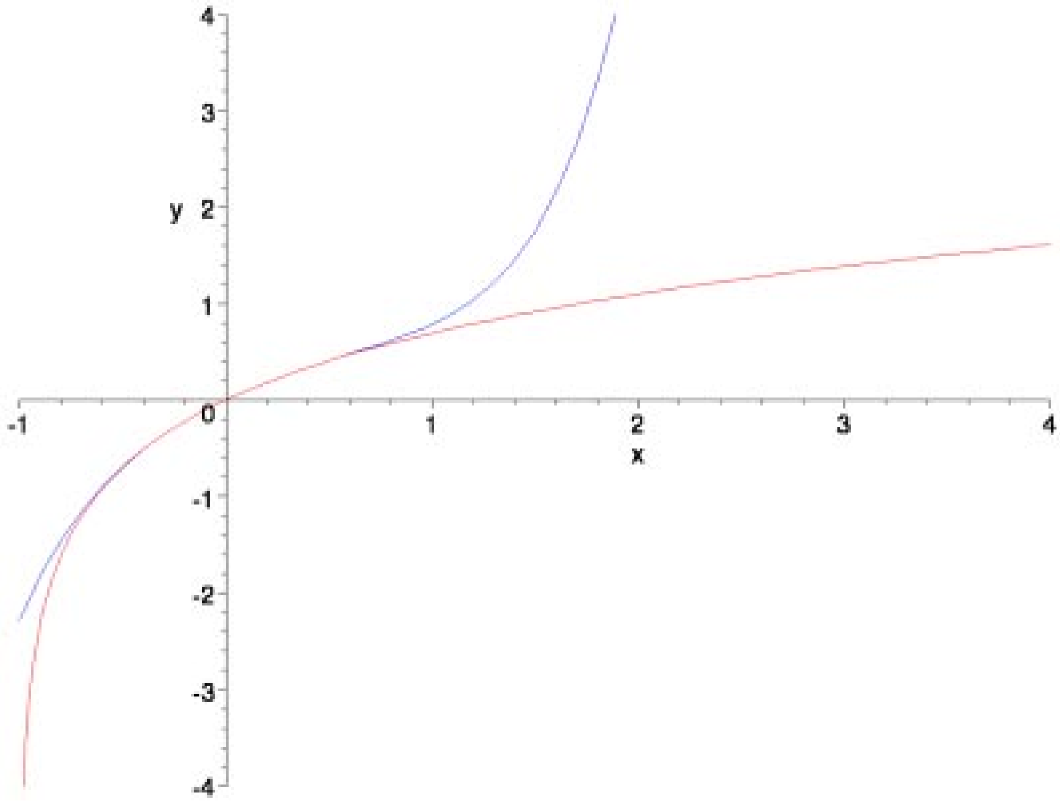
This looks good from -.7 to .9. For the fifth degree:

```
> T5:= taylor(f,x = 0,6);
> p5:=simplify(convert(T5,polynom));
```

$$T5 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + O(x^6)$$

$$p5 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

```
> plot0:=plot(f,x=-1..4,y=-4..4,color=red):
> plot1:=plot(p5,x=-1..4,y=-4..4,color=blue):
> display(plot0,plot1);
```



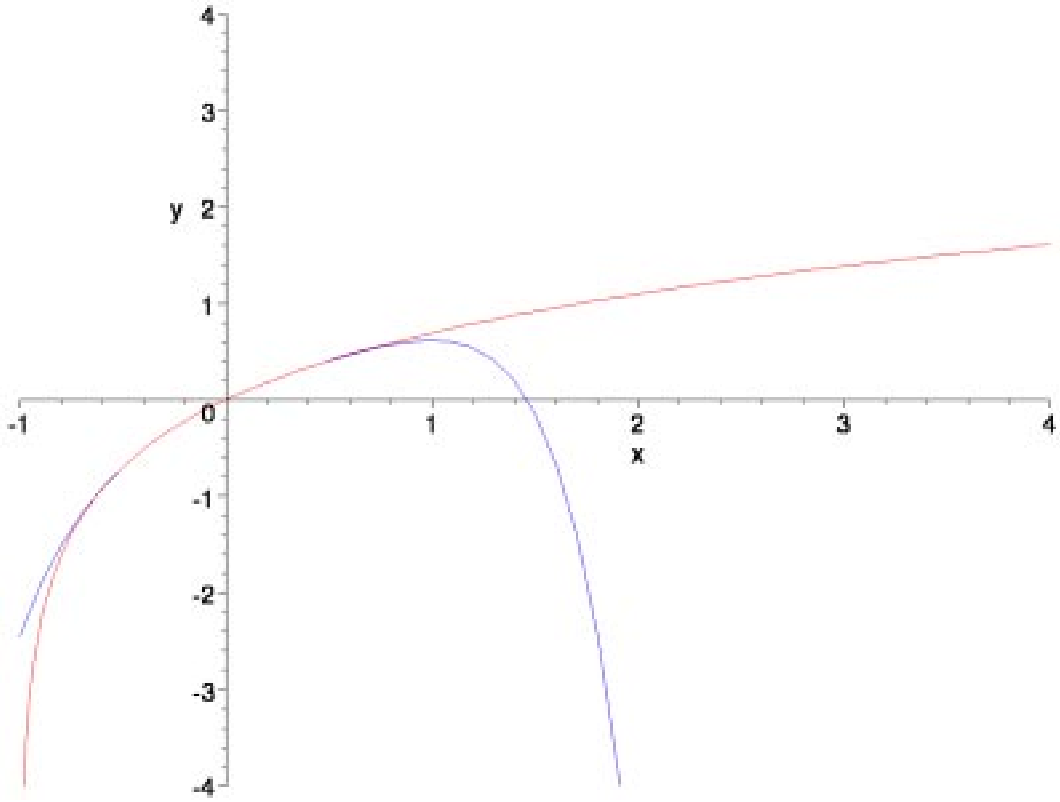
Now we get good approximations from -.7 to 1. For degree six:

```
> T6:= taylor(f,x = 0,7);
> p6:=simplify(convert(T6,polynom));
```

$$T6 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + O(x^7)$$

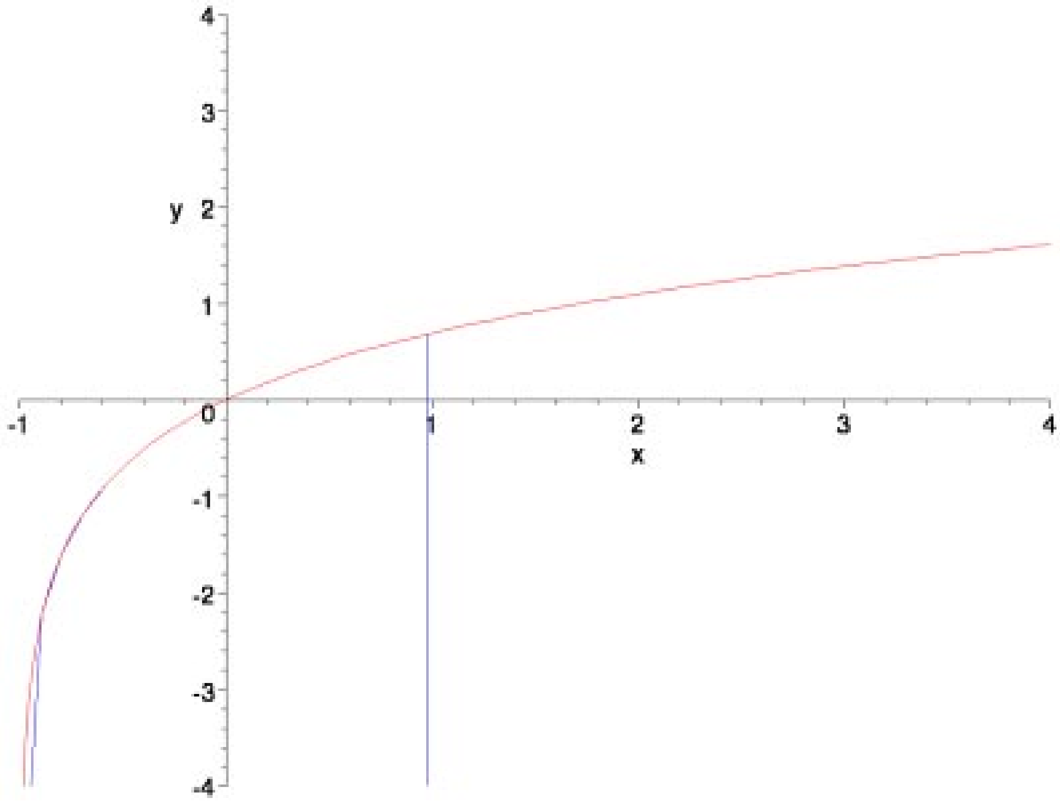
$$p6 := x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$$

```
> plot0:=plot(f,x=-1..4,y=-4..4,color=red):
> plot1:=plot(p6,x=-1..4,y=-4..4,color=blue):
> display(plot0,plot1);
```



This looks good from -.8 to .9. Now let's jump to degree 200:

```
> T200:= taylor(f,x = 0,201):
> p200:=simplify(convert(T200,polynom)):
> plot0:=plot(f,x=-1..4,y=-4..4,color=red):
> plot1:=plot(p200,x=-1..4,y=-4..4,color=blue):
> display(plot0,plot1);
```



With the following spreadsheet, we get a second view how well various Taylor polynomials approximate our function near 0.

```
> for i from 1 to 3 do
> T[10^i]:= taylor(f,x = 0,10^i+1):
> p[10^i]:=simplify(convert(T[10^i],polynom)):
> od:
```

A	B	C	D	E	F
x	$\ln(x+1)$	$P_4(x)$	$P_{10}(x)$	$P_{100}(x)$	$P_{1000}(x)$
-.5	-.6931471806	-.6822916667	-.6930648562	-.6931471808	-.6931471808
-.4	-.5108256238	-.5077333333	-.5108195900	-.5108256235	-.5108256235
-.3	-.3566749439	-.3560250000	-.3566747217	-.3566749441	-.3566749441
-.2	-.2231435513	-.2230666667	-.2231435491	-.2231435514	-.2231435514
-.1	-.1053605157	-.1053583333	-.1053605157	-.1053605156	-.1053605156
.1	.0953101798	.0953083333	.0953101798	.0953101798	.0953101798
.2	.1823215568	.1822666667	.1823215553	.1823215570	.1823215570
.3	.2623642645	.2619750000	.2623641381	.2623642645	.2623642645
.4	.3364722366	.3349333333	.3364694450	.3364722365	.3364722365
.5	.4054651081	.4010416667	.4054346478	.4054651080	.4054651080
.6	.4700036292	.4596000000	.4697906479	.4700036295	.4700036295
.7	.5306282511	.5093083333	.5295320470	.5306282510	.5306282510
.8	.5877866649	.5482666667	.5832754570	.5877866646	.5877866646
.9	.6418538862	.5739750000	.6261981044	.6418537612	.6418538863
1	.6931471806	.5833333333	.6456349206	.6881721793	.6926474306

>

Taylor Polynomials about x = a.

>

Now we look at the Taylor polynomial of degree n about $x = a$ for a function f. This is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^k(a)\,(x-a)^k.$$

This polynomial has the same value as the function at a, and it's first n derivatives have the same value at a as do the corresponding derivatives of the function. Let's examine these Taylor polynomials further by using the function $\mathrm{f}(x) = \sin(x)$ about $x = \frac{\pi}{6}$.

```
> restart:with(plots):
Warning, the name changecoords has been redefined
```

```
> f:=sin(x);
```

$$f := \sin(x)$$

We begin with degree 0.

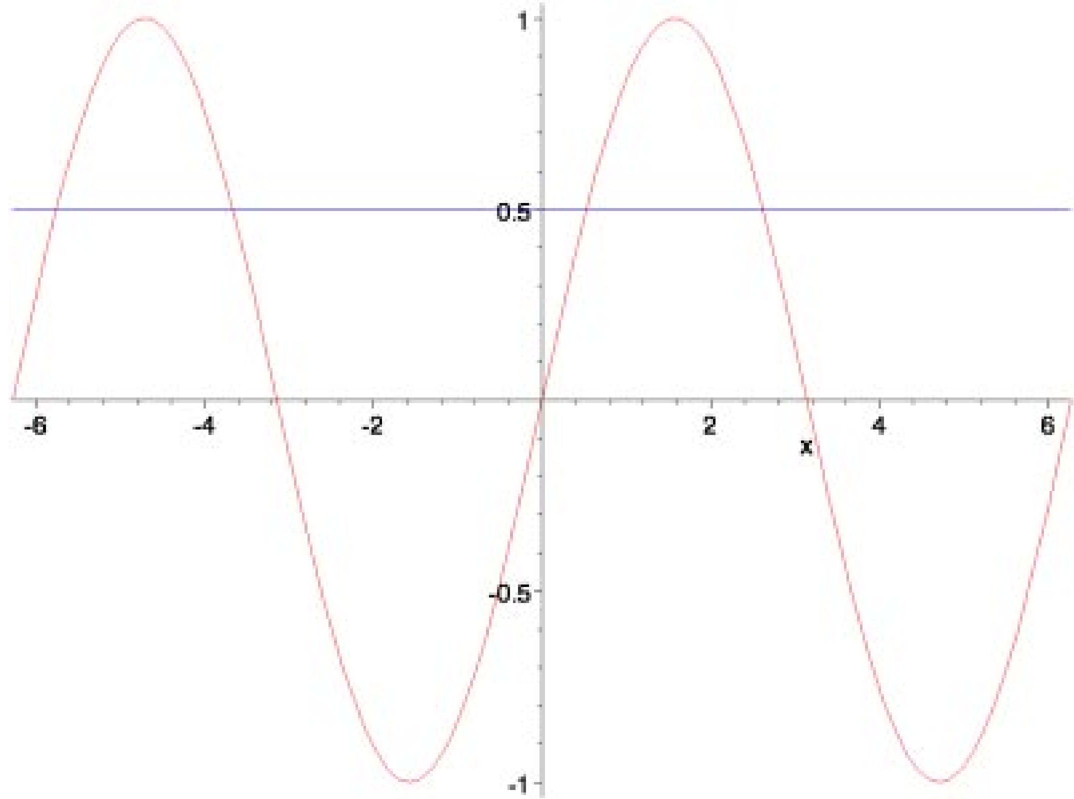
```
> T0:=taylor(f,x = Pi/6,1);
> p0 := convert(T0,polynom);
```

$$T0 := \frac{1}{2} + \mathrm{O}\left(x - \frac{1}{6}\pi\right)$$

$$p0 := \frac{1}{2}$$

We plot the function (in red) along with the Taylor polynomial (in blue).

```
> plot0:=plot(f,x=-2*Pi..2*Pi,color=red):
> plot1:=plot(p0,x=-2*Pi..2*Pi,color=blue):
> display(plot0,plot1);
```



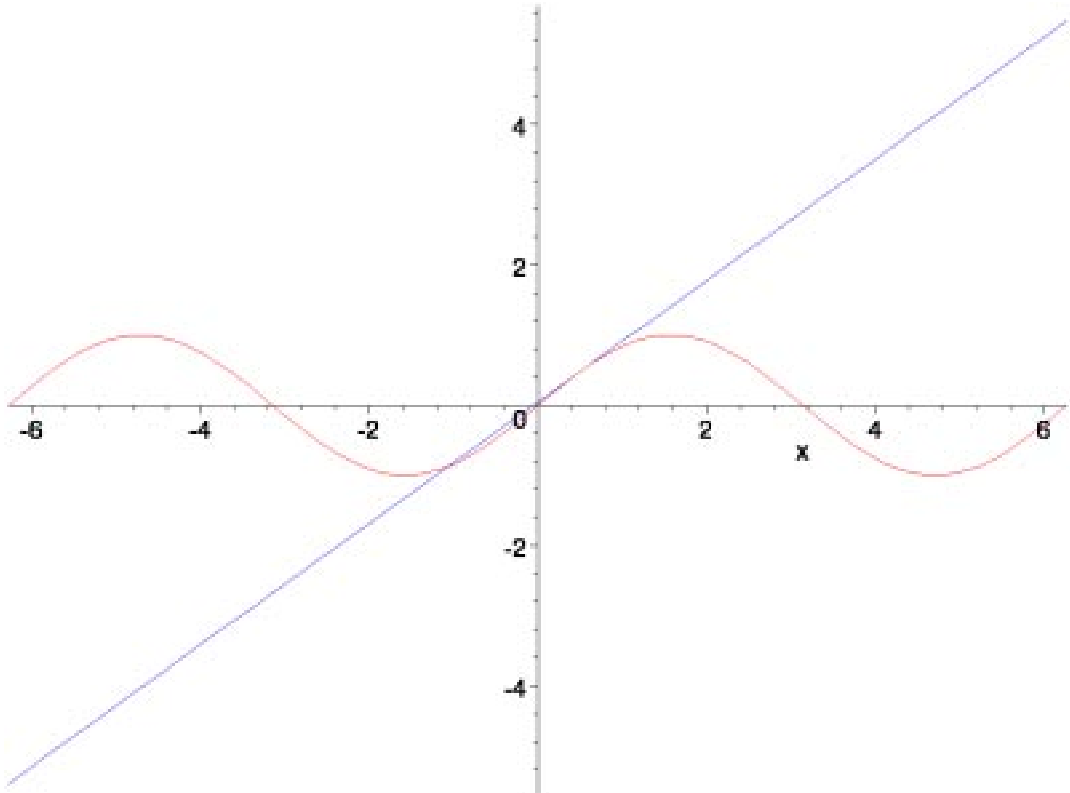
Degree 1 or linear approximation. Notice the use of collect to get a polynomial in *x*.

```
> T1:=taylor(f,x = Pi/6,2);
> p1 := collect(convert(T1,polynom),x);
```

$$T1 := \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) + \mathrm{O}\left(\left(x - \frac{1}{6}\pi\right)^2\right)$$

$$p1 := \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,color=red):
> plot1:=plot(p1,x=-2*Pi..2*Pi,color=blue):
> display(plot0,plot1);
```



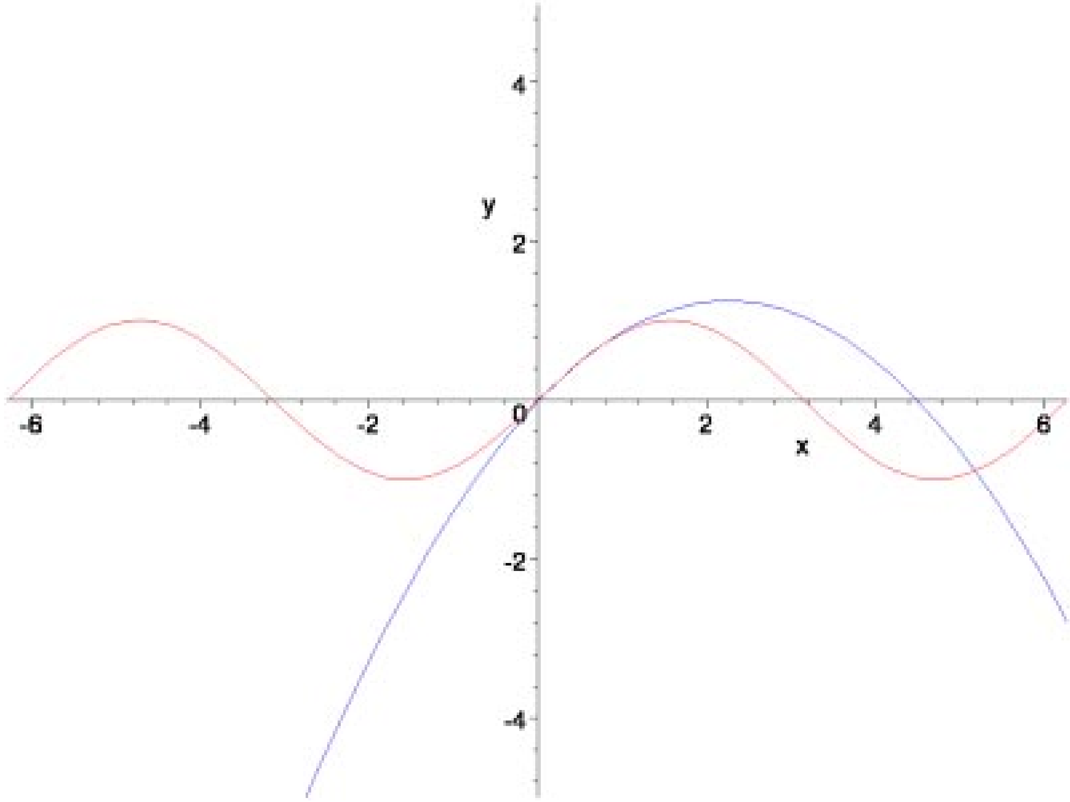
Second degree.

```
> T2:= taylor(f,x = Pi/6,3);
> p2:=collect(convert(T2,polynom),x);
```

$$T2 := \frac{1}{2}+\frac{1}{2}\sqrt{3}\left(x-\frac{1}{6}\pi\right)-\frac{1}{4}\left(x-\frac{1}{6}\pi\right)^2+\mathrm{O}\left(\left(x-\frac{1}{6}\pi\right)^3\right)$$

$$p2 := -\frac{1}{4}x^2+\left(\frac{1}{2}\sqrt{3}+\frac{1}{12}\pi\right)x+\frac{1}{2}-\frac{1}{12}\sqrt{3}\,\pi-\frac{1}{144}\pi^2$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,y=-5..5,color=red):
> plot1:=plot(p2,x=-2*Pi..2*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```



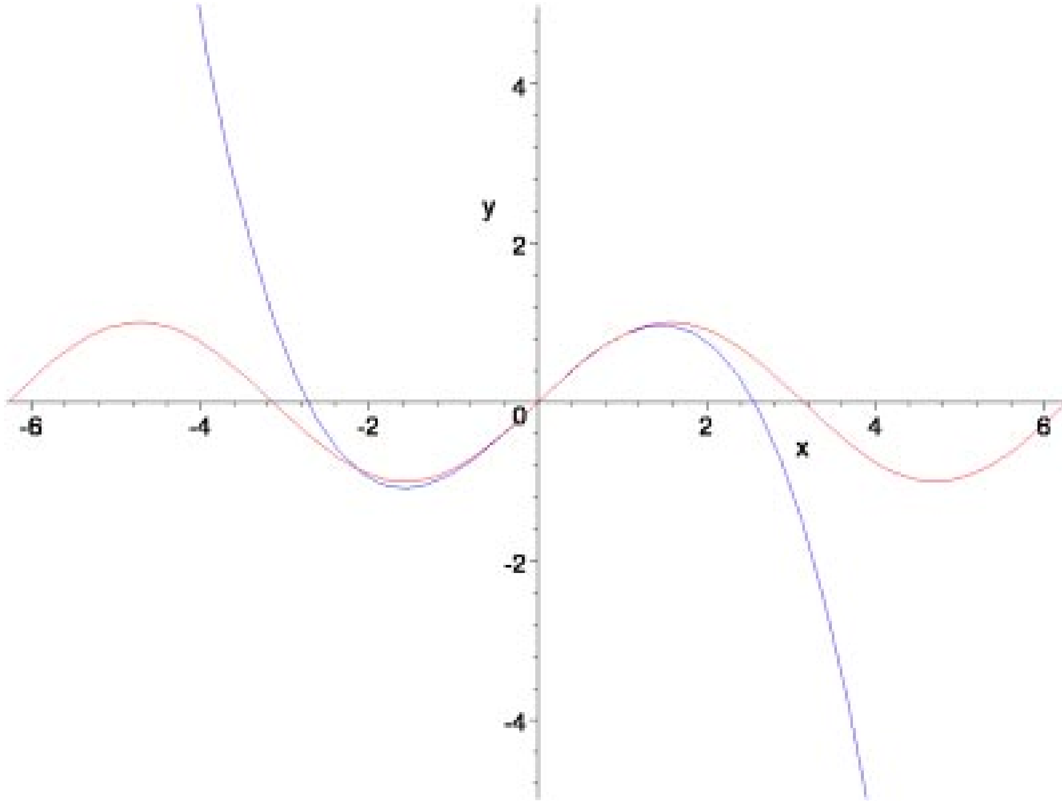
Degree 3.

```
> T3:= taylor(f,x = Pi/6,4);
> p3:=collect(convert(T3,polynom),x);
```

$$T3 := \frac{1}{2}+\frac{1}{2}\sqrt{3}\left(x-\frac{1}{6}\pi\right)-\frac{1}{4}\left(x-\frac{1}{6}\pi\right)^2-\frac{1}{12}\sqrt{3}\left(x-\frac{1}{6}\pi\right)^3+\mathrm{O}\left(\left(x-\frac{1}{6}\pi\right)^4\right)$$

$$p3 := -\frac{1}{12}\sqrt{3}\,x^3+\left(-\frac{1}{4}+\frac{1}{24}\sqrt{3}\,\pi\right)x^2+\left(\frac{1}{12}\pi+\frac{1}{2}\sqrt{3}-\frac{1}{144}\sqrt{3}\,\pi^2\right)x+\frac{1}{2}-\frac{1}{12}\sqrt{3}\,\pi$$
$$+\frac{1}{2592}\sqrt{3}\,\pi^3-\frac{1}{144}\pi^2$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,y=-5..5,color=red):
> plot1:=plot(p3,x=-2*Pi..2*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```



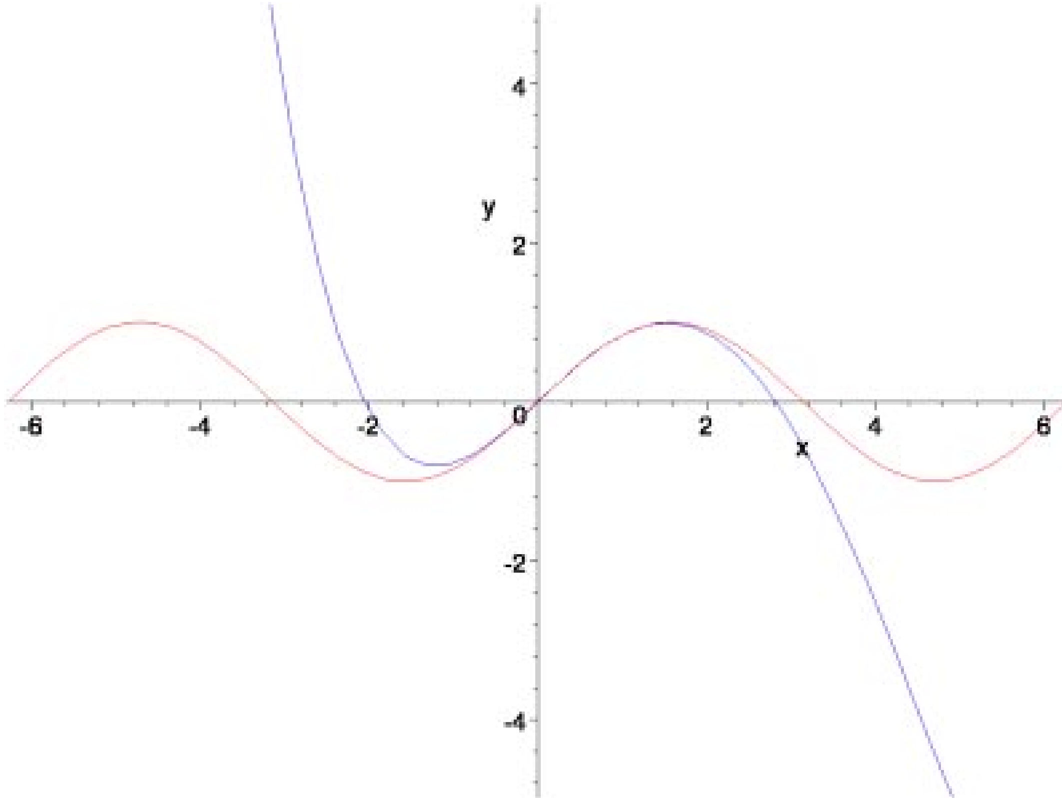
Degree 4.

```
> T4:= taylor(f,x = Pi/6,5);
> p4:=collect(convert(T4,polynom),x);
```

$$T4 := \frac{1}{2}+\frac{1}{2}\sqrt{3}\left(x-\frac{1}{6}\pi\right)-\frac{1}{4}\left(x-\frac{1}{6}\pi\right)^2-\frac{1}{12}\sqrt{3}\left(x-\frac{1}{6}\pi\right)^3+\frac{1}{48}\left(x-\frac{1}{6}\pi\right)^4+\mathrm{O}\left(\left(x-\frac{1}{6}\pi\right)^5\right)$$

$$p4 := \frac{1}{48}x^4+\left(-\frac{1}{12}\sqrt{3}-\frac{1}{72}\pi\right)x^3+\left(-\frac{1}{4}+\frac{1}{24}\sqrt{3}\,\pi+\frac{1}{288}\pi^2\right)x^2$$
$$+\left(\frac{1}{12}\pi-\frac{1}{144}\sqrt{3}\,\pi^2+\frac{1}{2}\sqrt{3}-\frac{1}{2592}\pi^3\right)x+\frac{1}{2}-\frac{1}{12}\sqrt{3}\,\pi+\frac{1}{2592}\sqrt{3}\,\pi^3-\frac{1}{144}\pi^2+\frac{1}{62208}\pi^4$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,y=-5..5,color=red):
> plot1:=plot(p4,x=-2*Pi..2*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```



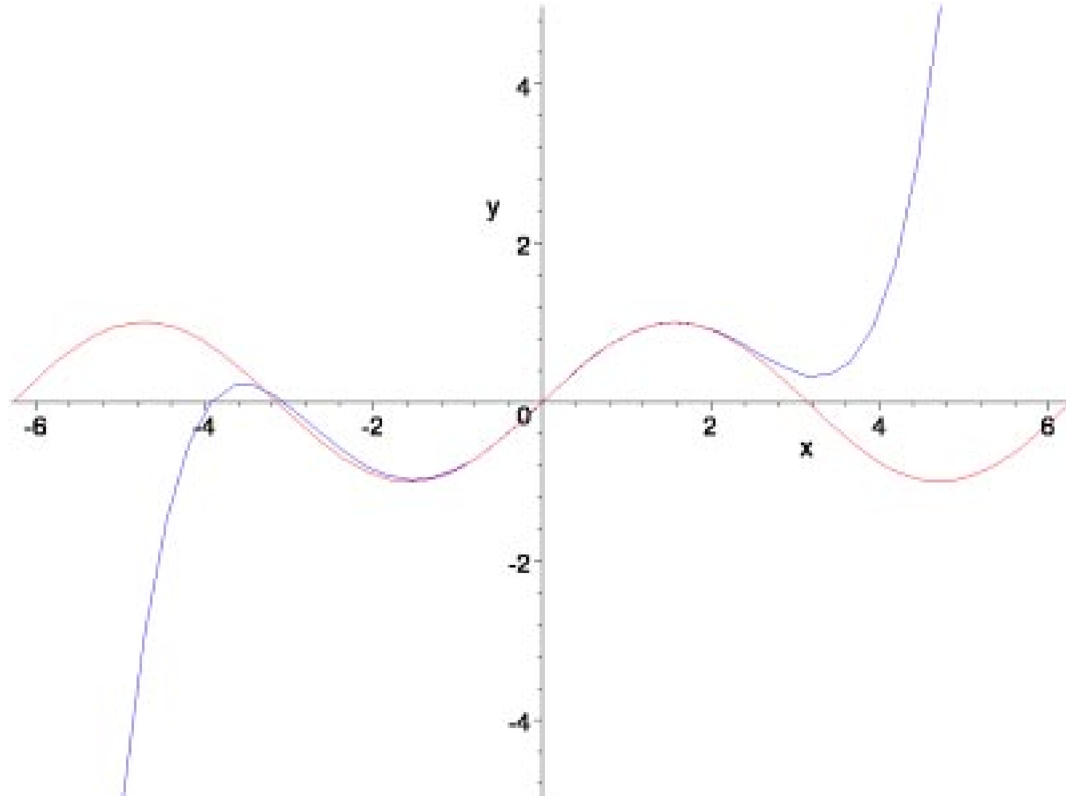
Degree 5:

```
> T5:= taylor(f,x = Pi/6,6);
> p5:=collect(convert(T5,polynom),x);
```

$$T5 := \frac{1}{2}+\frac{1}{2}\sqrt{3}\left(x-\frac{1}{6}\pi\right)-\frac{1}{4}\left(x-\frac{1}{6}\pi\right)^2-\frac{1}{12}\sqrt{3}\left(x-\frac{1}{6}\pi\right)^3+\frac{1}{48}\left(x-\frac{1}{6}\pi\right)^4+\frac{1}{240}\sqrt{3}\left(x-\frac{1}{6}\pi\right)^5+\mathrm{O}\left(\left(x-\frac{1}{6}\pi\right)^6\right)$$

$$p5 := \frac{1}{240}\sqrt{3}\,x^5+\left(\frac{1}{48}-\frac{1}{288}\sqrt{3}\,\pi\right)x^4+\left(-\frac{1}{12}\sqrt{3}+\frac{1}{864}\sqrt{3}\,\pi^2-\frac{1}{72}\pi\right)x^3+\left(-\frac{1}{4}-\frac{1}{5184}\sqrt{3}\,\pi^3+\frac{1}{24}\sqrt{3}\,\pi+\frac{1}{288}\pi^2\right)x^2+\left(\frac{1}{12}\pi-\frac{1}{144}\sqrt{3}\,\pi^2+\frac{1}{62208}\sqrt{3}\,\pi^4-\frac{1}{2592}\pi^3+\frac{1}{2}\sqrt{3}\right)x+\frac{1}{2}-\frac{1}{12}\sqrt{3}\,\pi-\frac{1}{1866240}\sqrt{3}\,\pi^5-\frac{1}{144}\pi^2+\frac{1}{2592}\sqrt{3}\,\pi^3+\frac{1}{62208}\pi^4$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,y=-5..5,color=red):
> plot1:=plot(p5,x=-2*Pi..2*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```



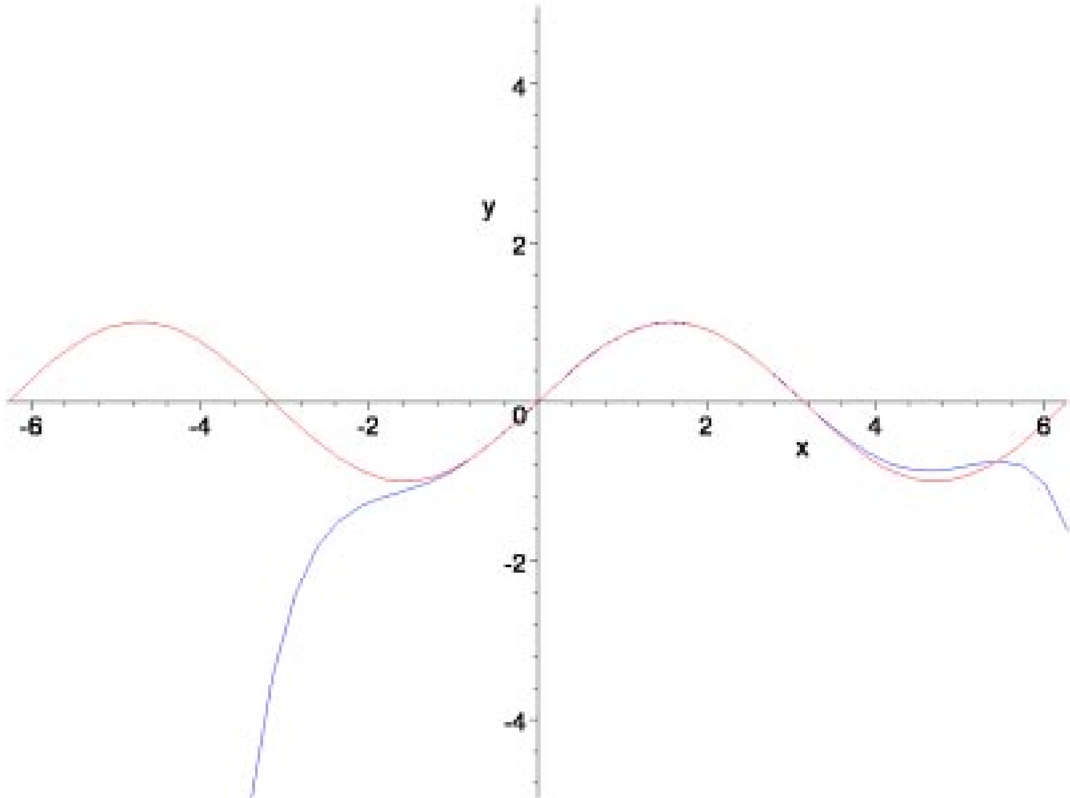
Degree 6.

```
> T6:= taylor(f,x = Pi/3,7);
> p6:=collect(convert(T6,polynom),x);
```

$$T6 := \frac{1}{2}\sqrt{3}+\frac{1}{2}\left(x-\frac{1}{3}\pi\right)-\frac{1}{4}\sqrt{3}\left(x-\frac{1}{3}\pi\right)^2-\frac{1}{12}\left(x-\frac{1}{3}\pi\right)^3+\frac{1}{48}\sqrt{3}\left(x-\frac{1}{3}\pi\right)^4+\frac{1}{240}\left(x-\frac{1}{3}\pi\right)^5-\frac{1}{1440}\sqrt{3}\left(x-\frac{1}{3}\pi\right)^6+\mathrm{O}\left(\left(x-\frac{1}{3}\pi\right)^7\right)$$

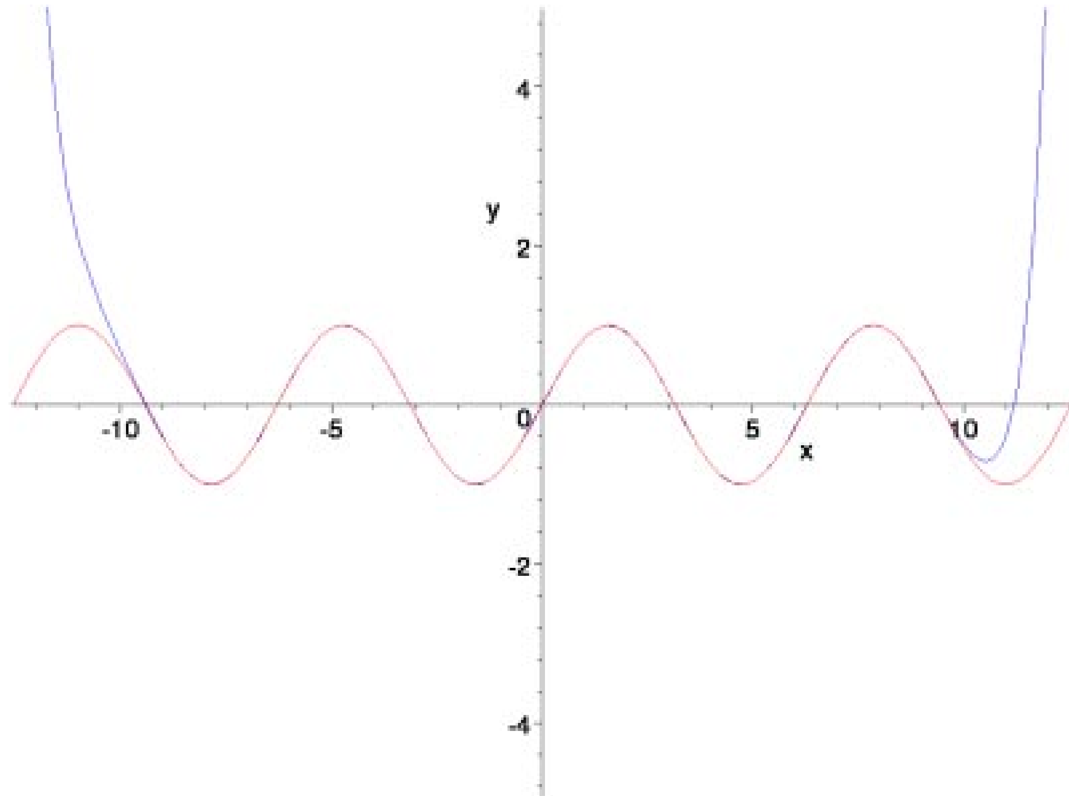
$$p6 := -\frac{1}{1440}\sqrt{3}\,x^6+\left(\frac{1}{240}+\frac{1}{720}\sqrt{3}\,\pi\right)x^5+\left(\frac{1}{48}\sqrt{3}-\frac{1}{144}\pi-\frac{1}{864}\sqrt{3}\,\pi^2\right)x^4$$
$$+\left(\frac{1}{216}\pi^2-\frac{1}{12}+\frac{1}{1944}\sqrt{3}\,\pi^3-\frac{1}{36}\sqrt{3}\,\pi\right)x^3$$
$$+\left(-\frac{1}{4}\sqrt{3}-\frac{1}{648}\pi^3+\frac{1}{72}\sqrt{3}\,\pi^2+\frac{1}{12}\pi-\frac{1}{7776}\sqrt{3}\,\pi^4\right)x^2$$
$$+\left(\frac{1}{2}+\frac{1}{58320}\sqrt{3}\,\pi^5+\frac{1}{6}\sqrt{3}\,\pi-\frac{1}{36}\pi^2+\frac{1}{3888}\pi^4-\frac{1}{324}\sqrt{3}\,\pi^3\right)x+\frac{1}{2}\sqrt{3}-\frac{1}{6}\pi-\frac{1}{36}\sqrt{3}\,\pi^2$$
$$-\frac{1}{1049760}\sqrt{3}\,\pi^6+\frac{1}{324}\pi^3-\frac{1}{58320}\pi^5+\frac{1}{3888}\sqrt{3}\,\pi^4$$

```
> plot0:=plot(f,x=-2*Pi..2*Pi,y=-5..5,color=red):
> plot1:=plot(p6,x=-2*Pi..2*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```



Degree 25.

```
> T25:= taylor(f,x = Pi/6,26):
> p25:=collect(convert(T25,polynom),x):
> plot0:=plot(f,x=-4*Pi..4*Pi,y=-5..5,color=red):
> plot1:=plot(p25,x=-4*Pi..4*Pi,y=-5..5,color=blue):
> display(plot0,plot1);
```


4
2
y
-10
-5
0
5
10
x
-2
-4

>
>
>
>