```
>
```

Chain Rule

```
> restart;
```

Derivative of a single variable.

We want to find $\frac{\partial}{\partial t} f$ where $f = \frac{x\,e^{z}}{y}$, $x = r\cos(t)$, $y = r\sin(t)$, and $z = \tan(t)$. We enter the four expressions and take the derivative.

```
> f:=x/y*exp(z);
```

$$f := \frac{x\,e^{z}}{y}$$

```
> x:=sin(t);
```

$$x := \sin(t)$$

```
> y:=cos(t);
```

$$y := \cos(t)$$

```
> z:=tan(t);
```

$$z := \tan(t)$$

```
> derivative:=diff(f,t);
```

$$derivative := e^{\tan(t)} + \frac{\sin(t)^2\,e^{\tan(t)}}{\cos(t)^2} + \frac{\sin(t)\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)}}{\cos(t)}$$

We simplify the derivative.

```
> derivative:=simplify(%,trig);
```

$$derivative := \frac{e^{\frac{\sin(t)}{\cos(t)}}\,\left(\sin(t) + \cos(t)\right)}{\cos(t)^3}$$

We can also find higher derivatives, for instance, the third derivative.

```
> third_derivative:=diff(f,t$3);
```

$$\begin{aligned}
third\_derivative := {} & 6\tan(t)\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)} + 3\,\left(1 + \tan(t)^2\right)^2\,e^{\tan(t)} + 2\,e^{\tan(t)} + \frac{8\sin(t)^2\,e^{\tan(t)}}{\cos(t)^2} \\
& + \frac{6\sin(t)\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)}}{\cos(t)} + \frac{6\sin(t)^4\,e^{\tan(t)}}{\cos(t)^4} + \frac{6\sin(t)^3\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)}}{\cos(t)^3} \\
& + \frac{6\sin(t)^2\tan(t)\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)}}{\cos(t)^2} + \frac{3\sin(t)^2\,\left(1 + \tan(t)^2\right)^2\,e^{\tan(t)}}{\cos(t)^2} \\
& + \frac{2\sin(t)\,\left(1 + \tan(t)^2\right)^2\,e^{\tan(t)}}{\cos(t)} + \frac{4\sin(t)\tan(t)^2\,\left(1 + \tan(t)^2\right)\,e^{\tan(t)}}{\cos(t)} \\
& + \frac{6\sin(t)\tan(t)\,\left(1 + \tan(t)^2\right)^2\,e^{\tan(t)}}{\cos(t)} + \frac{\sin(t)\,\left(1 + \tan(t)^2\right)^3\,e^{\tan(t)}}{\cos(t)}
\end{aligned}$$

This certainly needs simplifying.

```
> third_derivative:=simplify(%,trig);
```

$$third\_derivative := -\frac{e^{\frac{\sin(t)}{\cos(t)}}\left(-\sin(t)+4\sin(t)\cos(t)^4-18\sin(t)\cos(t)^2+4\cos(t)^5-9\cos(t)\right)}{\cos(t)^7}$$

You can also do the same thing using function notation, as follows.

```
> x:=t->sin(t);
```

$$x := t \rightarrow \sin(t)$$

```
> y:=t->cos(t);
```

$$y := t \rightarrow \cos(t)$$

```
> z:=t->tan(t);
```

$$z := t \rightarrow \tan(t)$$

```
> f:=(x,y)->x(t)/y(t)*exp(z(t));
```

$$f := (x, y) \rightarrow \frac{x(t)\, e^{z(t)}}{y(t)}$$

```
> derivative:=diff(f(x,y),t);
```

$$derivative := e^{\tan(t)} + \frac{\sin(t)^2 e^{\tan(t)}}{\cos(t)^2} + \frac{\sin(t)\left(1+\tan(t)^2\right) e^{\tan(t)}}{\cos(t)}$$

```
> derivative:=simplify(%,trig);
```

$$derivative := \frac{e^{\frac{\sin(t)}{\cos(t)}}\left(\sin(t)+\cos(t)\right)}{\cos(t)^3}$$

```
> third_derivative:=diff(f(x,y),t$3);
```

$$\begin{aligned}
third\_derivative :=\; & 6\tan(t)\left(1+\tan(t)^2\right)e^{\tan(t)} + 3\left(1+\tan(t)^2\right)^2 e^{\tan(t)} + 2\,e^{\tan(t)} + \frac{8\sin(t)^2 e^{\tan(t)}}{\cos(t)^2} \\
& + \frac{6\sin(t)\left(1+\tan(t)^2\right)e^{\tan(t)}}{\cos(t)} + \frac{6\sin(t)^4 e^{\tan(t)}}{\cos(t)^4} + \frac{6\sin(t)^3\left(1+\tan(t)^2\right)e^{\tan(t)}}{\cos(t)^3} \\
& + \frac{6\sin(t)^2\tan(t)\left(1+\tan(t)^2\right)e^{\tan(t)}}{\cos(t)^2} + \frac{3\sin(t)^2\left(1+\tan(t)^2\right)^2 e^{\tan(t)}}{\cos(t)^2} \\
& + \frac{2\sin(t)\left(1+\tan(t)^2\right)^2 e^{\tan(t)}}{\cos(t)} + \frac{4\sin(t)\tan(t)^2\left(1+\tan(t)^2\right)e^{\tan(t)}}{\cos(t)} \\
& + \frac{6\sin(t)\tan(t)\left(1+\tan(t)^2\right)^2 e^{\tan(t)}}{\cos(t)} + \frac{\sin(t)\left(1+\tan(t)^2\right)^3 e^{\tan(t)}}{\cos(t)}
\end{aligned}$$

```
> third_derivative:=simplify(%,trig);
```

$$third\_derivative := -\frac{e^{\frac{\sin(t)}{\cos(t)}}\left(-\sin(t)+4\sin(t)\cos(t)^4-18\sin(t)\cos(t)^2+4\cos(t)^5-9\cos(t)\right)}{\cos(t)^7}$$

Partial Derivatives.

The next example deals with converting from rectangular to polar coordinates. Suppose $z = 3\,y^2\,e^x$. We want to find all of the first and second order partial derivatives.

```
> restart;
> z:=3*y^2*exp(x);
```

$$z := 3\,y^2\,e^x$$

We enter the equations for the change of variables.

```
> x:=r*cos(theta);
```

$$x := r\cos(\theta)$$

```
> y:=r*sin(theta);
```

$$y := r\sin(\theta)$$

We find the first-order partial derivatives.

```
> Z[r]:=diff(z,r);
```

$$Z_r := 6\,r\sin(\theta)^2\,e^{r\cos(\theta)} + 3\,r^2\sin(\theta)^2\cos(\theta)\,e^{r\cos(\theta)}$$

```
> Z[theta]:=diff(z,theta);
```

$$Z_\theta := 6\,r^2\sin(\theta)\,e^{r\cos(\theta)}\cos(\theta) - 3\,r^3\sin(\theta)^3\,e^{r\cos(\theta)}$$

We find the second-order partial derivatives.

```
> Z_r_r:=diff(z,r$2);
```

$$Z\_r\_r := 6\sin(\theta)^2\,e^{r\cos(\theta)} + 12\,r\sin(\theta)^2\cos(\theta)\,e^{r\cos(\theta)} + 3\,r^2\sin(\theta)^2\cos(\theta)^2\,e^{r\cos(\theta)}$$

```
> Z_r_theta:=diff(diff(z,r),theta);
```

$$Z\_r\_theta := 12\,r\sin(\theta)\,e^{r\cos(\theta)}\cos(\theta) - 9\,r^2\sin(\theta)^3\,e^{r\cos(\theta)} + 6\,r^2\sin(\theta)\cos(\theta)^2\,e^{r\cos(\theta)} - 3\,r^3\sin(\theta)^3\cos(\theta)\,e^{r\cos(\theta)}$$

```
> Z_theta_theta:=diff(z,theta$2);
```

$$Z\_theta\_theta := 6\,r^2\cos(\theta)^2\,e^{r\cos(\theta)} - 15\,r^3\sin(\theta)^2\,e^{r\cos(\theta)}\cos(\theta) - 6\,r^2\sin(\theta)^2\,e^{r\cos(\theta)} + 3\,r^4\sin(\theta)^4\,e^{r\cos(\theta)}$$

Implicit Differentiation.

```
> restart;
```

The following equation defines *z* as an implicit function of *x* and *y*.

```
> eqn:=3*y*z^2-exp(4*x)*cos(4*z)-3*y^2=4;
```

$$eqn := 3\,y\,z^2 - e^{4x}\cos(4\,z) - 3\,y^2 = 4$$

We take the implicit partial derivatives of *z* with respect to *x* and *y* using the implicitdiff command.

```
> Z_x:=implicitdiff(eqn,z,x);
```

$$Z\_x := \frac{2\,e^{4x}\cos(4\,z)}{3\,y\,z + 2\,e^{4x}\sin(4\,z)}$$

```
> Z_y:=implicitdiff(eqn,z,y);
```

$$Z\_y := \frac{3}{2}\,\frac{-z^2 + 2\,y}{3\,y\,z + 2\,e^{4x}\sin(4\,z)}$$

```
>
```