Math 131 - Lab 2 (Fall 2005)
Limits and Continuity

Part One - Limits

Numerically

We write $\lim_{x\to c} f(x)$ to represent the number L approached by $f(x)$ as x approaches c. Note that $\lim_{x\to c} f(x)$ is read the limit of $f(x)$ as x approaches c. For instance, it makes sense to say that $\lim_{x\to 2}(2x+2) = 6$ since it seems clear that $f(x) = 2x+2$ gets close to 6 as x gets close to 2. But let's check this out numerically with our calculators. On the home page of the TI-86 and on the entry line of the home page of the TI-89 enter `{1.9,1.99,1.999,2.001,2.01,2.1} STO▸ x ENTER 2x+2 ENTER`. For the TI-86, the "{" and "}" are found under `2nd LIST`. Hit that key combination first. We have generated the information in the following table.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	5.8	5.98	5.998	6.002	6.02	6.2

It seems pretty clear that $f(x) = 2x+2$ does get close to 6 as x gets close to 2, approaching both from the left and the right.

In general, provided you get a number that makes sense, you can evaluate a limit like $\lim_{x\to 2}(2x+2)$ by simply plugging the 2 into the function formula to get 6. However, this would not work for $\lim_{x\to 3} \frac{x^2-4x+3}{x-3}$ since this would evaluate to $\frac{0}{0}$, which is not a number. Keeping in mind that $\lim_{x\to 3} \frac{x^2-4x+3}{x-3}$ and all other limits are only concerned with what happens as x gets close to 3, but not at 3 itself, and noting that $x-3 \neq 0$ when $x \neq 3$, we can cancel as in the following to get

$$\lim_{x\to 3} \frac{x^2-4x+3}{x-3} = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x-1) = 2$$

Because we are not concerned with the case $x = 3$, we are not canceling 0's when we cancel the $(x-3)$'s.

Do the following problems.

(a) Use your calculator as above to fill in the following table for $f(x) = \frac{\sin x}{x}$ (I suggest using 6 decimal places, but for sure at least 4).

x	$-.1$	$-.01$	$-.001$	.001	.01	.1
$f(x)$						

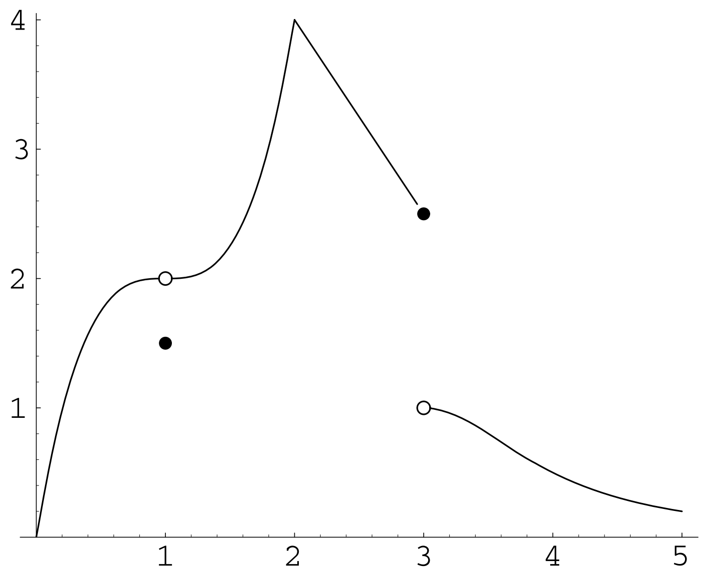
From the table, $\lim_{x\to 0} \frac{\sin x}{x} =$ ______.

(b) $\lim_{x\to 5} x^3 - 3x + 40 =$ ______.

(c) $\lim_{x \to -2} \frac{x^2 + 6x + 8}{x + 2} =$ ______.

Graphically

We first wish to graphically find several limits relating to the function whose graph is shown below. Note that $\lim_{x \to 1^-} f(x)$ is read the limit of $f(x)$ as x approaches 1 from the left and $\lim_{x \to 1^+} f(x)$ is read the limit of $f(x)$ as x approaches 1 from the right.



Find the value of each of the following from the graph. If the value does not exist (is undefined), write DNE.

(a) $\lim_{x \to 1^-} f(x) =$ ______

(b) $\lim_{x \to 1^+} f(x) =$ ______

(c) $\lim_{x \to 1} f(x) =$ ______

(d) $f(1) =$ ______

(e) $\lim_{x \to 2^-} f(x) =$ ______

(f) $\lim_{x \to 2^+} f(x) =$ ______

(g) $\lim_{x \to 2} f(x) =$ ______

(h) $f(2) =$ ______

(i) $\lim_{x \to 3^-} f(x) =$ ______

(j) $\lim_{x \to 3^+} f(x) =$ ______

(k) $\lim_{x \to 3} f(x) =$ ______

(l) $f(3) =$ ______

(m) $\lim_{x \to 4^-} f(x) =$ ______

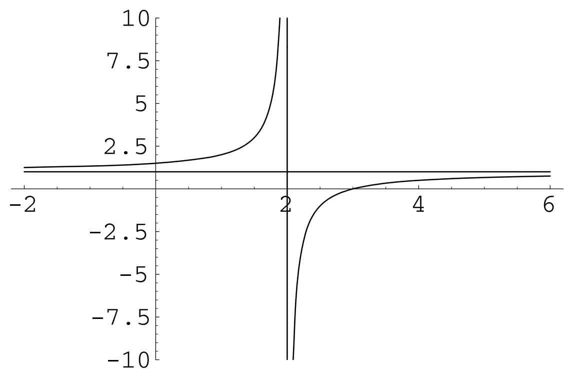
(n) $\lim_{x \to 4^+} f(x) =$ ______

(o) $\lim_{x \to 4} f(x) =$ ______

(p) $f(4) =$ ______

Algebraically

To find $\lim_{x \to c} f(x)$ algebraically, the basic rule we have been using is that you just substitute c for x in the function formula and evaluate, provided we get a meaningful number (e.g., the denominator is not 0). We are now going to expand on this. First, although $\lim_{x \to 2^-} (x-2) = \lim_{x \to 2^+} (x-2) = 0$, I will think of $\lim_{x \to 2^-} (x-2) = 0^-$, meaning that as x approaches 2 from the left, $(x-2)$ approaches 0 from the left, i.e., is negative and close to 0. Also, I will think of $\lim_{x\to 2^+}(x-2) = 0^+$, meaning that as x approaches 2 from the right, $(x-2)$ approaches 0 from the right, i.e., is positive and close to 0. With this in mind, we can then say that $\lim_{x\to 2^-}\frac{x-3}{x-2} = \frac{-1}{0^-} = +\infty$ and $\lim_{x\to 2^+}\frac{x-3}{x-2} = \frac{-1}{0^+} = -\infty$. (Note that this is still no help in the cases where we get something of the form $\frac{0}{0}$.) This explains the asymptotic behavior of the function $f(x) = \frac{x-3}{x-2}$ near $x = 2$ where we have included the asymptotic line $x = 2$ in the graph below. Note that $\lim_{x\to 2}\frac{x-3}{x-2}$ does not exist since $\lim_{x\to 2^-}\frac{x-3}{x-2} \neq \lim_{x\to 2^+}\frac{x-3}{x-2}$.



We also see that the included line $y = 1$ is a horizontal asymptote since

$$\lim_{x\to\pm\infty}\frac{x-3}{x-2} = \lim_{x\to\pm\infty}\frac{1-\frac{3}{x}}{1-\frac{2}{x}} = \frac{1-0}{1-0} = 1.$$

Evaluate the following limits. If a limit does not exist, write DNE. A good way to check your work is to graph the various functions on your calculator and also evaluate the limits graphically.

(a) $\lim_{x\to 2^-}\frac{x^2-1}{x-2} =$ ______

(b) $\lim_{x\to 2^+}\frac{x^2-1}{x-2} =$ ______

(c) $\lim_{x\to 2}\frac{x^2-1}{x-2} =$ ______

(d) $\lim_{x\to 0}\frac{x^4}{x^5+5x^2} =$ ______

(e) $\lim_{x\to 1}\frac{(x-3)(x-1)}{(x-1)(x+2)} =$ ______

(f) $\lim_{x\to 3^-}\frac{(x+1)(x-3)}{x(x-3)^2} =$ ______

(g) $\lim_{x\to 3^+}\frac{(x+1)(x-3)}{x(x-3)^2} =$ ______

(h) $\lim_{x\to 3}\frac{(x+1)(x-3)}{x(x-3)^2} =$ ______

(i) $\lim_{x\to 0^-}\frac{|x|}{x} =$ ______

(j) $\lim_{x\to 0^+}\frac{|x|}{x} =$ ______

(k) $\lim_{x\to 0}\frac{|x|}{x} =$ ______

(l) $\lim_{x\to +\infty}\frac{3x^2+1}{x^2+1} =$ ______

(m) $\lim_{x \to -\infty} \frac{6x^2+1}{3x^2+7} =$ ______

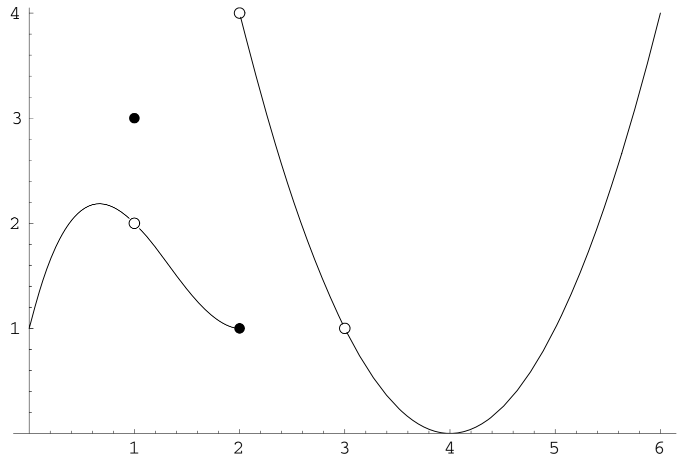
(n) $\lim_{x \to +\infty} \frac{x^4+3x+1}{7x^3+1} =$ ______

(o) $\lim_{x \to -\infty} \frac{x^4+3x+1}{7x^3+1} =$ ______

(p) $\lim_{x \to +\infty} \frac{3e^x+2}{2e^x+3} =$ ______

Part Two - Continuity

If you look at the graph of the function below, you can probably say exactly where the function f is and is not continuous.



1. For what values of x is f not continuous?

2. For each of the following values of c, find $\lim_{x \to c} f(x)$ (write DNE if the limit does not exist) and $f(c)$, and then state whether f is continuous at c.

   (a) c=1

   $\lim_{x \to 1} f(x) =$ ______

   $f(1) =$ ______

   Is f continuous at $x = 1$? ______

   (b) c=2

   $\lim_{x \to 2} f(x) =$ ______

   $f(2) =$ ______

   Is f continuous at $x = 2$? ______

   (c) c=3

   $\lim_{x \to 3} f(x) =$ ______

   $f(3) =$ ______

   Is f continuous at $x = 3$? ______

   (d) c=5

   $\lim_{x \to 5} f(x) =$ ______

   $f(5) =$ ______

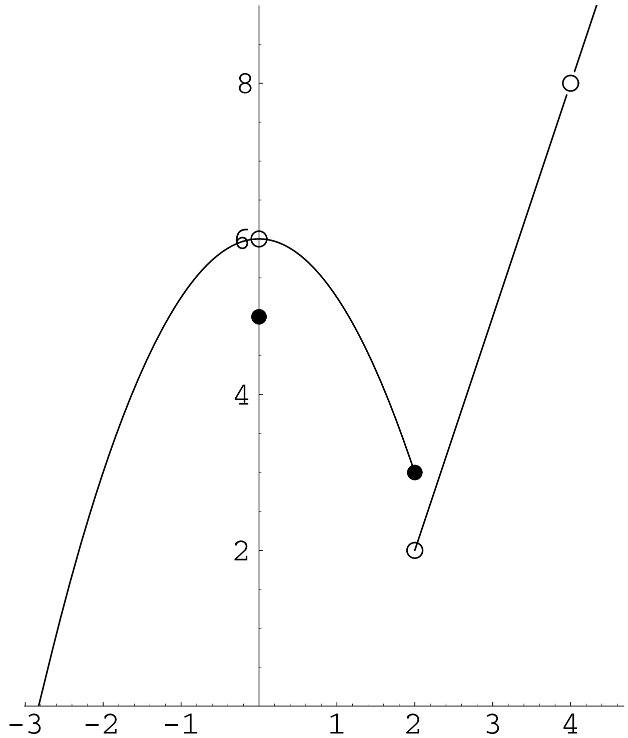
   Is f continuous at $x = 5$? ______

3. Complete the following definition based on your observations above:

   The function f is continuous at $x = c$ if

4. The discontinuities at $x = 1$ and $x = 3$ are called removable, while the discontinuity at $x = 2$ is not removable. From your observations, what determines whether a discontinuity is removable? Why do you think the word “removable” is used? Explain.

Limit Exercises



Find the value of each of the following from the graph. If the value does not exist (is undefined), write DNE.

(1) $\lim_{x\to 0^-} f(x) =$ ______ (b) $\lim_{x\to 0^+} f(x) =$ ______

(3) $\lim_{x\to 0} f(x) =$ ______ (4) $f(0) =$ ______

(5) $\lim_{x\to 1^-} f(x) =$ ______ (6) $\lim_{x\to 1^+} f(x) =$ ______

(7) $\lim_{x\to 1} f(x) =$ ______ (8) $f(1) =$ ______

(9) $\lim_{x\to 2^-} f(x) =$ ______ (10) $\lim_{x\to 2^+} f(x) =$ ______

(11) $\lim_{x\to 2} f(x) =$ ______ (12) $f(2) =$ ______

(13) $\lim_{x\to 4^-} f(x) =$ ______ (14) $\lim_{x\to 4^+} f(x) =$ ______

(15) $\lim_{x\to 4} f(x) =$ ______ (16) $f(4) =$ ______

Find the value of each of the following limits. If the value does not exist (is undefined), write DNE.

(17) $\lim_{x\to 2}(2x+3) =$ ______ (18) $\lim_{x\to -8} x^{1/3} =$ ______

(19) $\lim_{x\to -4} \sqrt{\dfrac{x+8}{25-x^2}} =$ ______ (20) $\lim_{x\to -5} |3x-2| =$ ______

(21) $\lim_{x\to 2^-} \dfrac{2(x+4)(x-1)}{(x+3)(x-2)} =$ ______ (22) $\lim_{x\to 2^+} \dfrac{2(x+4)(x-1)}{(x+3)(x-2)} =$ ______

(23) $\lim\limits_{x\to+\infty} \dfrac{2x^2+6x+8}{x^2+x-6} =$ ______

(24) $\lim\limits_{x\to-\infty} \dfrac{2(x+4)(x-1)}{(x+3)(x-2)} =$ ______

(25) $\lim\limits_{x\to+\infty} \dfrac{x+3}{2-x} =$ ______

(26) $\lim\limits_{x\to+\infty} \dfrac{x^2+4}{x+3} =$ ______

(27) $\lim\limits_{x\to-\infty} \dfrac{x^4+3x}{x^4+2x^5} =$ ______

(28) $\lim\limits_{x\to+\infty} \dfrac{2e^{-x}+3}{3e^{-x}+2} =$ ______

(29) $\lim\limits_{x\to 2} \dfrac{x^2+4x+4}{x+2} =$ ______

(30) $\lim\limits_{x\to 4^-} \dfrac{|x-4|}{x-4} =$ ______

(31) $\lim\limits_{x\to 4^+} \dfrac{|x-4|}{x-4} =$ ______

(32) $\lim\limits_{x\to 4} \dfrac{|x-4|}{x-4} =$ ______

Use your calculator to find the value of the following limits. If the value does not exist (is undefined), write DNE.

(33) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} =$ ______

(34) $\lim\limits_{x\to 0} \dfrac{x^2}{\sin x} =$ ______

(35) $\lim\limits_{x\to 0} \dfrac{2x}{\sin x - x} =$ ______

(36) $\lim\limits_{x\to 0} \dfrac{1}{x} \sin \dfrac{x}{3} =$ ______