CHAPTER 14

Vector Calculus

1. Vector Fields

Definition. A <u>vector field</u> in the plane is a function $\mathbf{F}(x,y)$ from $\mathbb{R}^2$ into V_2, We write

$$\mathbf{F}(x,y) = \langle f_1(x,y), f_2(x,y)\rangle = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j}.$$

A <u>vector field</u> in space is a function $\mathbf{F}(x,y,z)$ from $\mathbb{R}^3$ into V_3, We write

$$\mathbf{F}(x,y,z) = \langle f_1(x,y), f_2(x,y), f_3(x,y)\rangle = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} + f_3(x,y)\mathbf{k}.$$

Example. $\mathbf{F}(x,y) = \langle -y, x\rangle = -y\mathbf{i} + x\mathbf{j}$

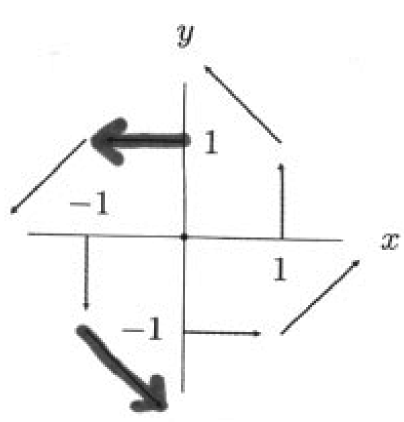
Some values:

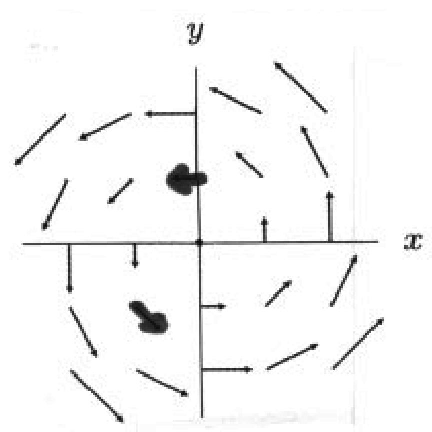
$\mathbf{F}(0,1) = \langle -1, 0\rangle = -\mathbf{i} \Longrightarrow \|\mathbf{F}(0,1)\| = 1.$

$\mathbf{F}(-1,-1) = \langle 1, -1\rangle = \mathbf{i} - \mathbf{j} \Longrightarrow \|\mathbf{F}(-1,-1)\| = \sqrt{2}.$

In general,

$\|\mathbf{F}(x,y)\| = \|\langle -y, x\rangle\| = \sqrt{x^2+y^2}.$





The diagram on the left on the previous page is drawn to scale, with $\mathbf{F}(x, y)$ placed at (x, y). The diagram on the right is scaled smaller with relative magnitudes to fit in the diagram.

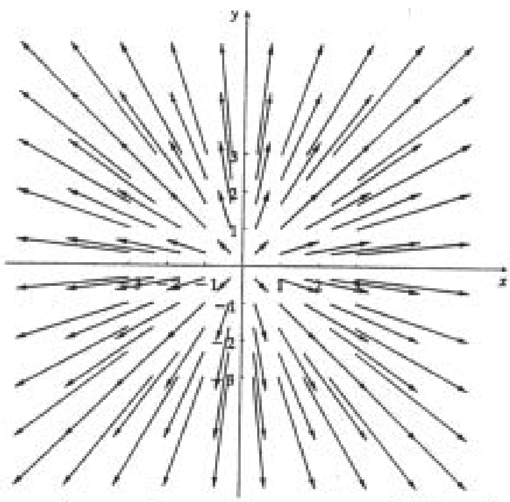
This vector field could be called a "spin" field. Since

$$\langle x, y\rangle \cdot \langle -y, x\rangle = -xy + xy = 0,$$

each $\mathbf{F}(x, y)$ is tangent to the circle centered at the origin of radius $\sqrt{x^2 + y^2}$, pointing in a counter-clockwise direction with magnitude equalling the radius of the circle.

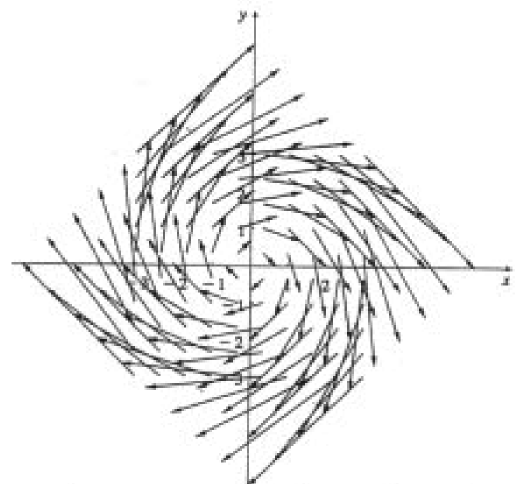
Sample vector fields:

1. $x\mathbf{i} + y\mathbf{j}$



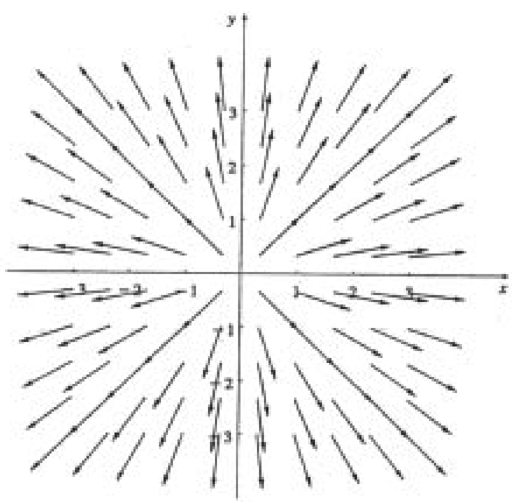
each vector has magnitude equal to the distance from its base to the origin

2. $y\mathbf{i} - x\mathbf{j}$



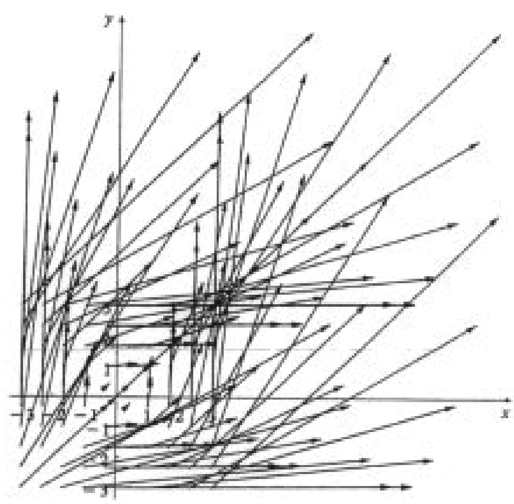
tangent to circle of radius $\sqrt{x^2+y^2}$, clockwise, magnitude = radius

3. $\dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{1/2}}$



same as above, except each vector is unit vector

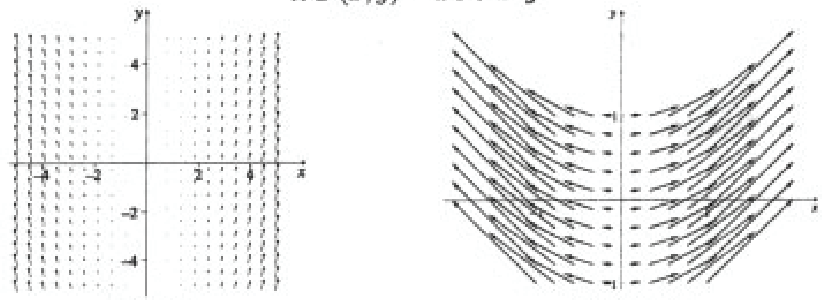
4. $y^2\mathbf{i} + x^2\mathbf{j}$



point on x-axis, vector points up; point on y-axis, vector points right; elsewhere, vector points to some point in 1st quadrant

More examples of scaled and unscaled vector fields:

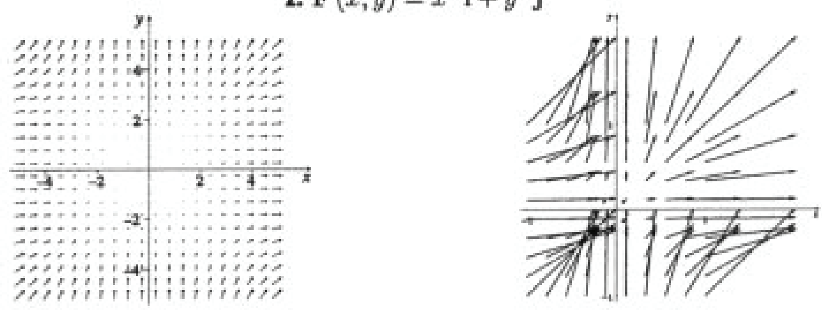
1. $\mathbf{F}(x, y) = x\,\mathbf{i} + x^2\,\mathbf{j}$



Scaled Unscaled

no downward motion here

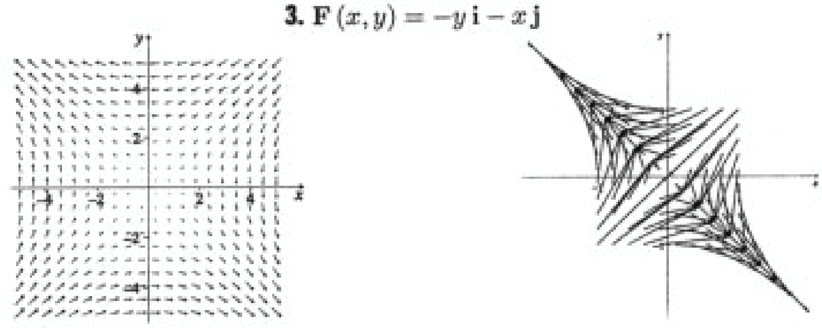
2. $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$



Scaled Unscaled

no motion downward or to the left

3. $\mathbf{F}(x, y) = -y\,\mathbf{i} - x\,\mathbf{j}$



Scaled Unscaled

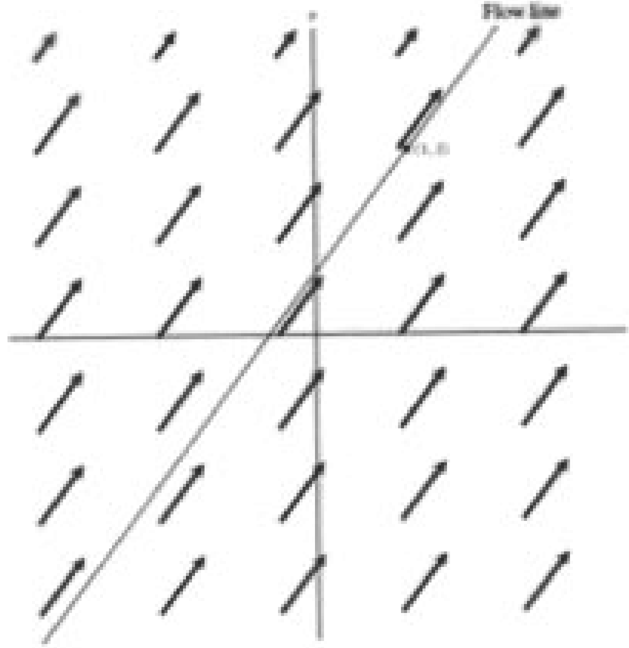
motion above x-axis is to the xxxxxxxxx Left, below to the xxxxxx Right

motion to right of y-axis is downward, to the left upward

Some vector fields are velocity vector fields, i.e., $\mathbf{F}(x, y)$ gives the velocity of a particle at (x, y). Suppose a particle starts to flow at (x_0, y_0) at time t_0. Then the curve traced out by $\langle x(t), y(t)\rangle$, where $x(t)$ and $y(t)$ are solutions of the differential equations

$$x'(t) = f_1\big((x(t), y(t)\big) \text{ and } y'(t) = f_2\big((x(t), y(t)\big)$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, is a flow line.



Vector field $\vec{F} = 3\vec{i} + 4\vec{j}$ with a flow line through $(1, 2)$

We use the chain rule to find a differential equation for y as a function of x for the velocity vector field:

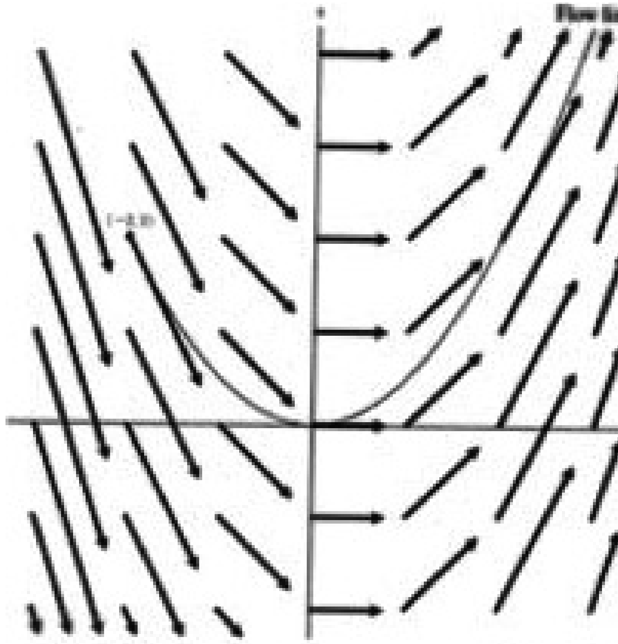
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{f_2(x, y)}{f_1(x, y)}.$$

In our example,

$$\frac{dy}{dx} = \frac{4}{3} \Longrightarrow y = \frac{4}{3}x + C.$$

For the flow line through $(1, 2)$, $2 = \dfrac{4}{3} + C \Longrightarrow C = \dfrac{2}{3}$. Thus $y = \dfrac{4}{3}x + \dfrac{2}{3}$ is the equation of the flow line.

EXAMPLE. We consider the vector field $\mathbf{F}(x,y) = \langle 1, x\rangle$ and the flow line through $(-2,2)$.



A flow line of the vector field $\vec{v} = \vec{i} + x\vec{j}$

This is a velocity vector field for $\dfrac{dy}{dx} = \dfrac{x}{1} = x$. Then $y = \dfrac{1}{2}x^2 + C$. For the flow line through $(-2,2)$, $2 = 2 + C \implies C = 0$. Thus the equation of the flow line is $y = \dfrac{1}{2}x^2$.

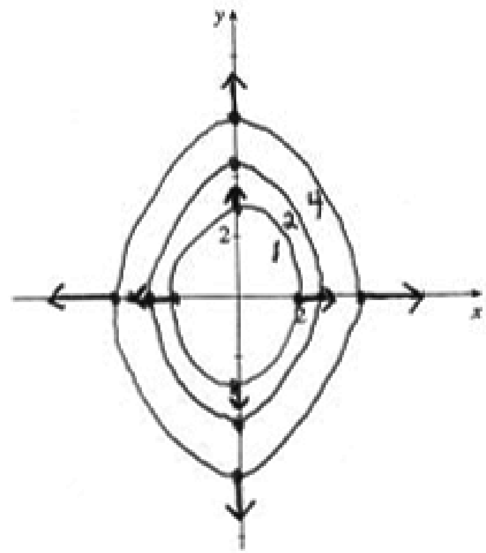
DEFINITION. For any scalar function f (from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}$), the vector field $\mathbf{F}(x,y) = \nabla f$ is called the <u>gradient field</u> for the function f. We call f a <u>potential function</u> for $\mathbf{F}$. Whenever $\mathbf{F} = \nabla f$ for some scalar function f, we refer to $\mathbf{F}$ as a <u>conservative vector field</u>.

Gradient Fields and Level Curves

Compute the gradient fields for the following functions, and draw level curves $f(x,y)=k$ for the indicated values of k. Then sketch the gradient vector field at one or two points on each of these level curves.

1. $f(x,y)=\frac{x^2}{4}+\frac{y^2}{9}$; $k=1,2,4$

$\nabla f(x,y)=\left(\frac{x}{2},\frac{2y}{9}\right)$



$k=1$: $\frac{x^2}{4}+\frac{y^2}{9}=1$

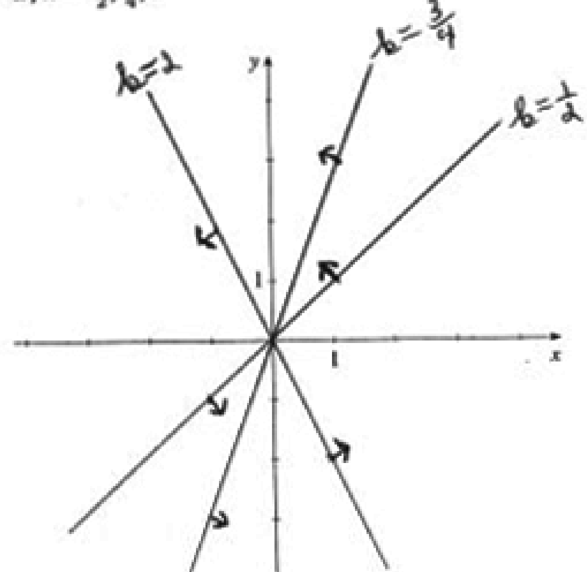
$k=2$: $\frac{x^2}{8}+\frac{y^2}{18}=1$

$k=4$: $\frac{x^2}{16}+\frac{y^2}{36}=1$

2. $f(x,y)=\frac{y}{x+y}$, $y\neq -x$; $k=\frac{1}{2},\frac{3}{4},2$

$\nabla f(x,y)=$

$\left(-\frac{y}{(x+y)^2},\frac{x}{(x+y)^2}\right)$



$k=\frac{1}{2}$: $y=x$

$k=\frac{3}{4}$: $y=3x$

$k=2$: $y=-2x$

In general, the further from the origin, the shorter the vector.

PROBLEM (Page 1128 #30). Determine whether

$$\mathbf{F}(x,y) = \langle y\cos x, \sin x - y\rangle$$

is conservative, and if so, find its potential function.

(We proceed in a manner different from the text.) This is the same as determining whether

$$\underbrace{y\cos x}_{M}\,dx + \underbrace{\sin x - y}_{N}\,dy = 0$$

is exact and finding its solution as we did in Math 231.

$$M_y = \cos x \text{ and } N_x = \cos x \Longrightarrow M_y = N_x \Longrightarrow$$
$$\text{the DE is exact} \Longrightarrow \mathbf{F} \text{ is conservative.}$$

$$\begin{aligned} f(x,y) &= \int y\cos x\,dx \\ &= y\sin x + g(y) \\ f_y(x,y) &= \sin x + g'(y) \\ &= \sin x - y \Longrightarrow \\ g'(y) = -y &\Longrightarrow g(y) = -\frac{y^2}{2} + C \end{aligned}$$

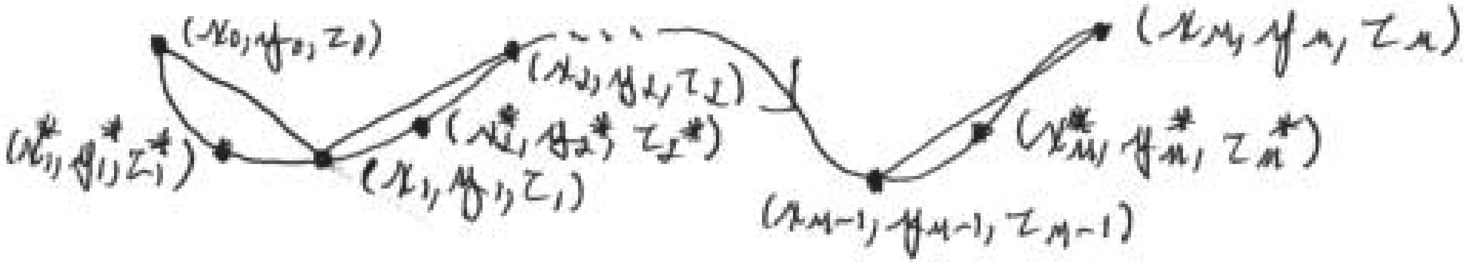
$$\begin{aligned} f(x,y) &= \int (\sin x - y)\,dy \\ &= y\sin x - \frac{y^2}{2} + h(x) \\ f_x(x,y) &= y\cos x + h'(x) \\ &= y\cos x \Longrightarrow \\ h'(x) = 0 &\Longrightarrow h(x) = C \end{aligned}$$

Thus

$$f(x,y) = y\sin x - \frac{y^2}{2} + C$$

2. Line Integrals

Oriented curve – one from which we have chosen a direction – two possible directions.



DEFINITION. The line integral of $f(x, y, z)$ with respect to arc length along the oriented curve C in three-dimensional space, written $\int f(x, y, z)\, ds$, is defined by

$$\int_C f(x, y, z)\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i,$$

provided the limit exists and is the same for all choices of evaluation points.

NOTE. There is a similar definition for two dimensions.

THEOREM (Evaluation Theorem). *Suppose that $f(x,y,z)$ is continuous in a region D containing the curve C and that C is described parametrically by $\big(x(t),y(t),z(t)\big)$ for $a \le t \le b$ where $x(t)$, $y(t)$, and $z(t)$ all have continuous first derivatives. Then*

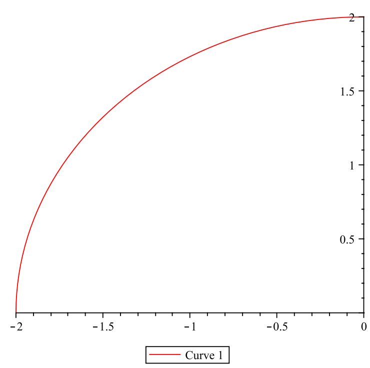
$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt.$$

In the two-dimensional case,

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$$

DEFINITION. A space curve C is smooth if it can be described parametrically by $x=x(t)$, $y=y(t)$, and $z=z(t)$ for $a\le t\le b$, where $x(t)$, $y(t)$, and $z(t)$ all have continuous first derivatives and $[x'(t)]^2+[y'(t)]^2+[z'(t)]^2\ne 0$ on $[a,b]$. (Similarly for plane curves.)

PROBLEM (Page 1143 #10A). Find $\displaystyle\int_C 3y^2\,ds$ where C is the quarter-circle $x^2+y^2=4$ from $(0,2)$ to $(-2,0)$.



$$x=2\cos t \implies x'(t)=-2\sin t \text{ and } y=2\sin t \implies y'(t)=2\cos t.$$

$$\int_C 3y^2\,ds = \int_{\pi/2}^{\pi}\big[3(4\sin^2 t)\big]\sqrt{4\sin^2 t+4\cos^2 t}\,dt = 24\int_{\pi/2}^{\pi}\sin^2 t\,dt =$$

$$12\int_{\pi/2}^{\pi}(1-\cos 2t)\,dt = 12\Big[t-\frac{1}{2}\sin 2t\Big]_{\pi/2}^{\pi} = 12\Big[\Big(\pi-0\Big)-\Big(\frac{\pi}{2}-0\Big)\Big]=6\pi.$$

PROBLEM (Page 1143 # 22). Find $\int_C xz\,ds$, where C is the line segment from $(2,1,0)$ to $(2,0,2)$.

$x = 2,$ and for $0 \le t \le 1$,

$$y = 1 - t, z = 2t \Longrightarrow x'(t) = 0, y'(t) = -1, z'(t) = 2.$$

$$ds = \sqrt{0^2 + (-1)^2 + 2}\,dt = \sqrt{5}\,dt$$

Thus

$$\int_C xz\,ds = \int_0^1 2(2t)\sqrt{5}\,dt = 4\sqrt{5}\int_0^1 t\,dt = 4\sqrt{5}\Big[\frac{t^2}{2}\Big]_0^1 = 4\sqrt{5}\Big(\frac{1}{2}-0\Big) = 2\sqrt{5}.$$

THEOREM. *Suppose $f(x,y,z)$ is a continuous function in some region D containing the oriented curve C. Then, if C is piecewise smooth, with $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_1, C_2, $\ldots$, C_n are all smooth and where the terminal point of C_i is the same as the initial point of C_{i+1}, for $i = 1, 2, \ldots, n-1$, we have*

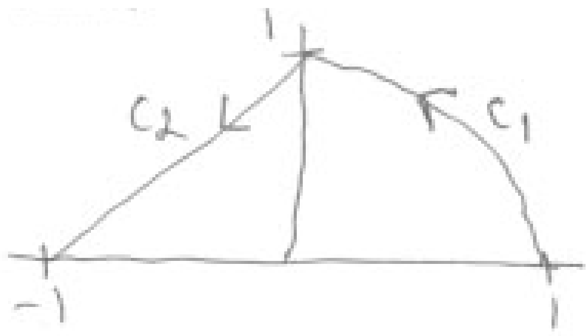
$$\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$$

and

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \cdots \int_{C_n} f(x,y,z)\,ds.$$

NOTE. There is a similar statement for two dimensions.

EXAMPLE. Find $\int_C (x+y)\,ds$ over $C = C_1 \cup C_2$ where C_1 is the quarter circle from $(1,0)$ to $(0,1)$ and C_2 is the line segment from $(0,1)$ to $(-1,0)$.



C_1: $x(t) = \cos t, y(t) = \sin t, 0 \le t \le \dfrac{\pi}{2} \Longrightarrow x'(t) = -\sin t, y'(t) = \cos t.$

C_2: $x(t) = -t, y(t) = 1-t, 0 \le t \le 1, x'(t) = -1, y'(t) = -1.$

Thus

$$\int_C (x+y)\,ds = \int_{C_1} (x+y)\,ds + \int_{C_2} (x+y)\,ds =$$

$$\int_0^{\pi/2} (\cos t + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt + \int_0^1 \bigl[-t + (1-t)\bigr]\sqrt{(-1)^2 + (-1)^2}\,dt =$$

$$\int_0^{\pi/2} (\cos t + \sin t)\,dt + \sqrt{2}\int_0^1 (1-2t)\,dt =$$

$$\Bigl[\sin t - \cos t\Bigr]_0^{\pi/2} + \sqrt{2}\Bigl[t - t^2\Bigr]_0^1 = 1 + 1 + \sqrt{2}(0-0) = 2.$$

THEOREM. *For any piecewise smooth curve C (in two or three dimensions), $\int_C 1\,ds$ gives the arc length of the curve C.*

<u>Line integrals with respect to x</u>

$$\int_C f(x,y,z)\,dx = \lim_{\|p\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta x_i$$

$$\int_C f(x,y,z)\,dx = \int_a^b f\big(x(t), y(t), z(t)\big)x'(t)\,dt$$

$$\int_{-C} f(x,y,z)\,dx = -\int_C f(x,y,z)\,dx$$

$$\int_C f(x,y,z)\,dx = \int_{C_1} f(x,y,z)\,dx + \int_{C_2} f(x,y,z)\,dx + \cdots \int_{C_n} f(x,y,z)\,dx$$

NOTE. We have similar results for y and z and two dimensions.

NOTATION. We write

$$\int_C f(x,y,z)\,dx + \int_C g(x,y,z)\,dy + \int_C h(x,y,z)\,dz = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$

EXAMPLE. Find $\int_C 3y^2\,dy$ where C is the line segment from $(2,0)$ to $(1,3)$.

Parameterize the line by $x = 2-t, y = 3t, 0 \le t \le 1$. Then $dy = 3\,dt$ and

$$\int_C 3y^2\,dy = \int_0^1 3(3t)^2(3dt) = 81\int_0^1 t^2\,dt = 27t^3\Big|_0^1 = 27.$$

PROBLEM (Page 1143 #16). Find $\int_C 3y^2\,dy$ where C is the portion of $y = x^2$ from $(2,4)$ to $(0,0)$.

1) Parameterize by $x = -t$, $y = t^2$ from $-2 \le t \le 0$. Then $dy = 2t\,dt$ and

$$\int_C 3y^2\,dy = \int_{-2}^{0} 3t^4(2t)\,dt = 6\int_{-2}^{0} t^5\,dt = t^6\Big|_{-2}^{0} = 0 - (-2)^6 = -64$$

2) Could also parameterize by $x = 2 - t$, $y = (2-t)^2$ from $0 \le t \le 2$. Then $dy = -2(2-t)\,dt$ and

$$\int_C 3y^2\,dy = \int_0^2 3(2-t)^4(-2)(2-t)\,dt = \\ -6\int_0^2 (2-t)^5\,dt = -(2-t)^6\Big|_0^2 = 0 - 64 = -64.$$

Line Integrals of Vector Fields

Let $\mathbf{F}(x,y,z) = \big\langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\big\rangle$ be a vector field along the curve C defined by $x = x(t)$, $y = y(t)$, and $z = z(t)$, $a \le t \le b$. Let

$$\mathbf{r} = \langle x, y, z\rangle \implies d\mathbf{r} = \langle dx, dy, dz\rangle.$$

Define the line integral

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz = \\ \int_C F_1(x,y,z)\,dx + \int_C F_2(x,y,z)\,dy + \int_C F_3(x,y,z)\,dz$$

Work

If $\mathbf{F}(x,y,z)$ is a force field, the work done by $\mathbf{F}$ in moving a particle along the curve C can be written as

$$W = \int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}.$$

PROBLEM (Page 1143 #36). Find the work done by $\mathbf{F}(x,y,x) = \langle z, 0, 3x^2\rangle$ along C, the quarter ellipse given by $x = 2\cos t$, $y = 3\sin t$, $z = 1$, from $(2,0,1)$ to $(0,3,1)$.

We have

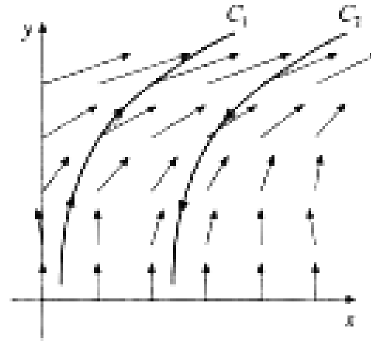
$$0 \le t \le \frac{\pi}{2}, \quad dx = -2\sin t\,dt, \quad dy = 3\cos t\,dt, \quad dz = 0.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C z\,dx + 0\,dy + F_3(x,y,z)\,dz =$$

$$\int_0^{\pi/2} \Big[(1)(-2\sin t) + 0(3\cos t) + 3(4\cos^2 t)(0)\Big]\,dt =$$

$$2\cos t\Big|_0^{\pi/2} = 0 - 2 = -2.$$

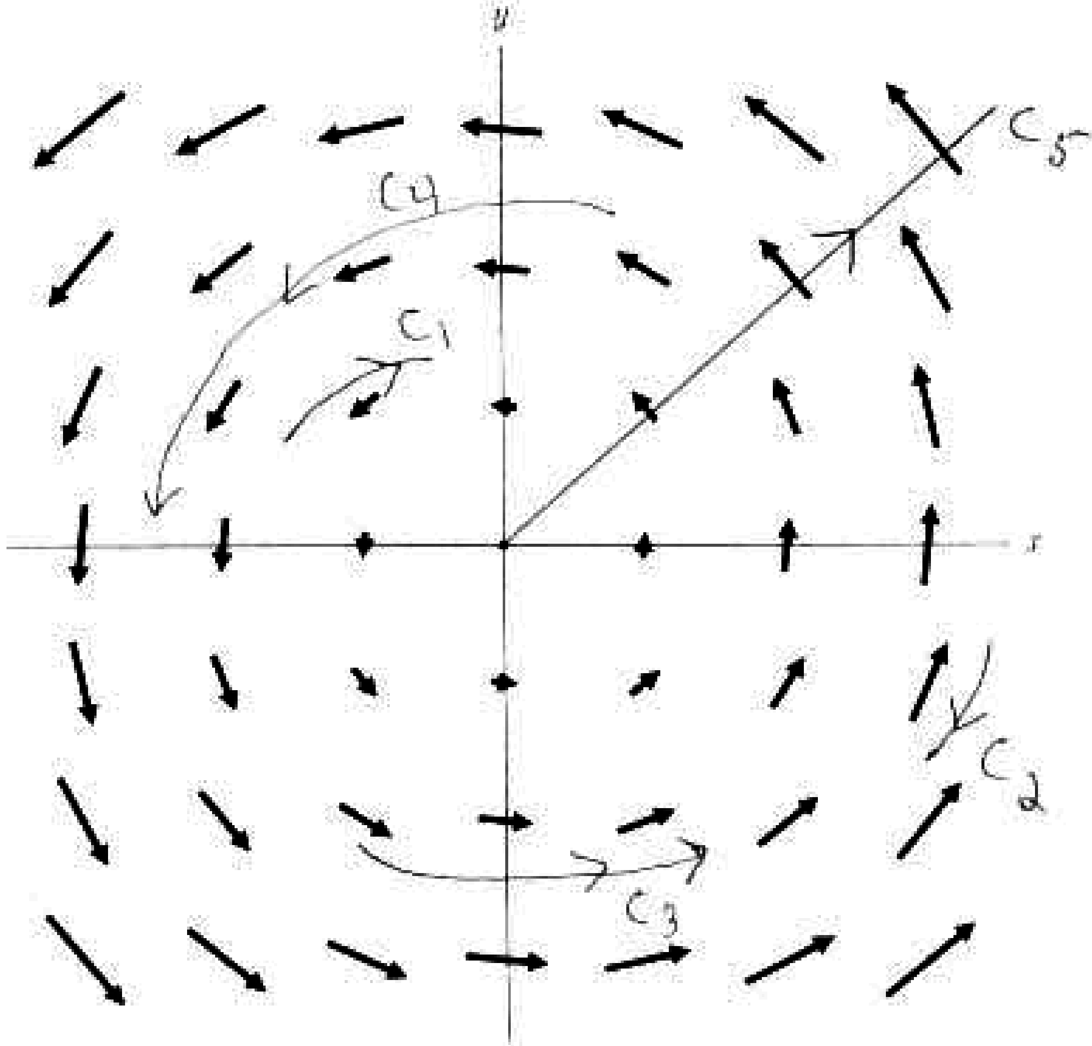
- Consider the vector field $\mathbf{F}(x,y)$ and the curves C_1 and C_2 shown below. Explain why $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} < 0$.



- Explain why $\int_C \mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_C \mathbf{G}\cdot d\mathbf{r} < 0$ in the diagram below.

Recall $\mathbf{F}\cdot d\mathbf{r} = \|\mathbf{F}\|\cdot\|d\mathbf{r}\|\cos\theta$, where θ is the angle between $\mathbf{F}$ and $d\mathbf{r}$, $0 \le \theta \le \pi$. Thus the line integral of a vector field measures the extent to which C is going with the vector field (+) or against it (−).

Using the diagram below, arrange $\int_{C_i} \mathbf{F} \cdot d\mathbf{r}$, $i = 1 \ldots 4$, and the number 0 in order from left to right (smallest to largest).



$$\vec{F}(x, y) = -y\vec{i} + x\vec{j}$$

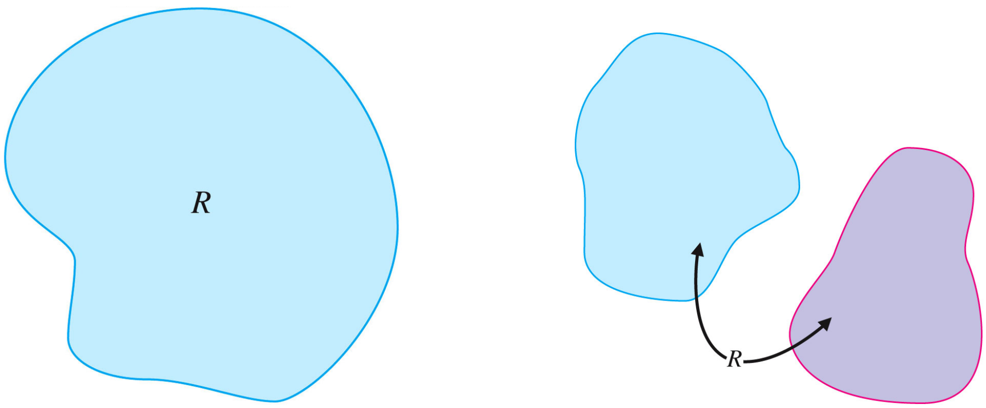
$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} < \int_{C_1} \mathbf{F} \cdot d\mathbf{r} < 0 < \int_{C_3} \mathbf{F} \cdot d\mathbf{r} < \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

C_1 and C_2 have the opposite direction of the vector field with the vectors on C_2 having the greater magnitude. C_3 and C_4 have the same direction of the vector field with the vectors having a similar magnitude, but C_4 is longer.

3. Independence of Path and Conservative Vector Fields

DEFINITION.

(1) A region $D \subseteq \mathbb{R}^n$ (for $n \geq 2$) is called connected if every pair of points in D can be connected by a piecewise-smooth curve lying entirely in D.



connected not connected

(2) A path is a piecewise-smooth curve C traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$ for $a \leq t \leq b$.

(3) The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in the domain D if the integral is the same for every path contained in D that has the same beginning and end points.

THEOREM. *Suppose the vector field* $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ *is continuous on the open, connected region* $D \subseteq \mathbb{R}^2$. *Then the line integral* $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ *is independent of path if and only if* $\mathbf{F}$ *is conservative on* D.

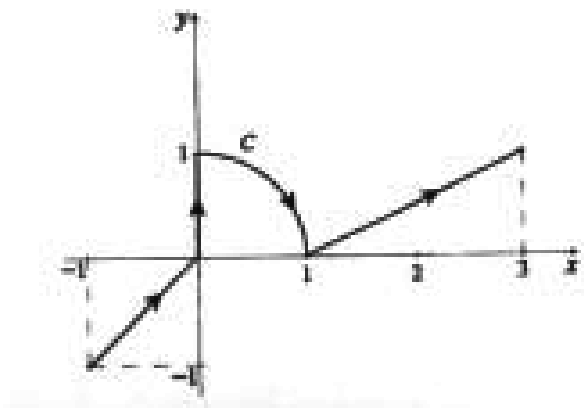
NOTE. A similar result is valid in any number of dimensions.

THEOREM (Fundamental Theorem for Line Integrals).

Suppose $\mathbf{F}(x,y) = \big\langle M(x,y), N(x,y)\big\rangle$ *is continuous on the open, connected region* $D \subseteq \mathbb{R}^2$ *and* C *is any piecewise-smooth curve lying in* D*, with initial point* (x_1, y_1) *and terminal point* (x_2, y_2)*. Then, if* $\mathbf{F}$ *is conservative on* D*, with* $\mathbf{F}(x,y) = \nabla f(x,y)$*,*

$$\int_C \mathbf{F}(x,y)\cdot d\mathbf{r} = f(x,y)\Big|_{(x_1,y_1)}^{(x_2,y_2)} = f(x_2,y_2) - f(x_1,y_1).$$

EXAMPLE. Compute $\displaystyle\int_C \langle ye^{xy}, xe^{xy}\rangle \cdot d\mathbf{r}$ for the curve C shown below.



We need to find $f(x,y)$ such that

$$\nabla f = \langle f_x, f_y\rangle = \langle ye^{xy}, xe^{xy}\rangle.$$

Assume $f_x = ye^{xy}$. Then

$$f(x,y) = \int ye^{xy}\,dx = e^{xy} + g(y) \Longrightarrow f_y(x,y) = xe^{xy} + g'(y).$$

Then $\nabla f = \mathbf{F}$ if $g(y) = C$ for some constant C. Choose $C = 0 \Longrightarrow$

$$f(x,y) = e^{xy}.$$

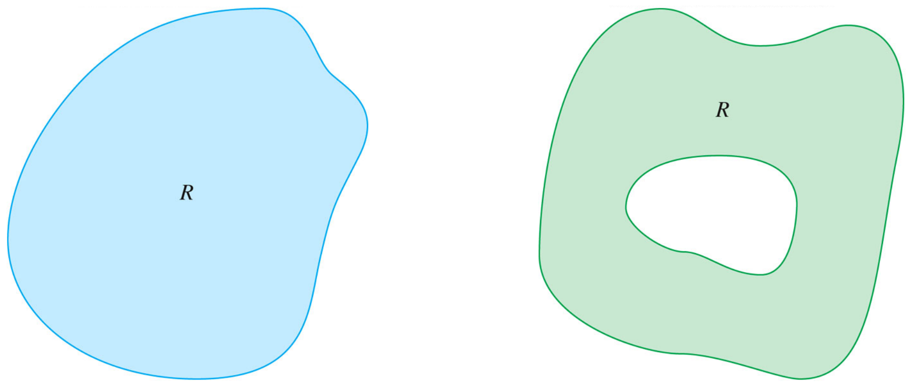
Then $\mathbf{F}$ is conservative on $\mathbb{R}^2$, so

$$\int_C \langle ye^{xy}, xe^{xy}\rangle \cdot d\mathbf{r} = e^{xy}\Big|_{(-1,-1)}^{(3,1)} = e^3 - e.$$

DEFINITION. A curve C is closed if its two endpoints are the same. For C defined by $x = g(t), y = h(t), a \leq t \leq b$, this means $\big(g(a), h(a)\big) = \big(g(b), h(b)\big)$.

THEOREM. *Suppose* $\mathbf{F}$ *is continuous on the open, connected region* $D \subseteq \mathbb{R}^2$. *Then* $\mathbf{F}$ *is conservative on* D *if and only if* $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ *for every piecewise-smooth closed curve* C *lying in* D.

DEFINITION. A region D is simply-connected if every closed curve in D encloses only points in D.



simply-connected not simply-connected

THEOREM. *Suppose* $M(x, y)$ *and* $N(x, y)$ *have continuous first partial derivatives on a simply-connected region* D. *Then* $\int_C M(x, y)\,dx + N(x, y)\,dy$ *is independent of path in* D *if and only if* $M_y(x, y) = N_x(x, y)$ *for all* (x, y) *in* D.

EXAMPLE. $\mathbf{F}(x, y) = \big\langle M(x, y), N(x, y)\big\rangle = \langle x - y, x - 2\rangle$. $M_y = -1$ and $N_x = 1$. Thus

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C (x - y)\,dx + (x - 2)\,dy$$

is not independent of path.

THEOREM (Conservative Vector Fields). *Suppose* $\mathbf{F}(x, y) = \left\langle M(x, y), N(x, y)\right\rangle$ *and* $M(x, y)$ *and* $N(x, y)$ *have continuous first partial derivatives on an open, simply-connected region* $D \subseteq \mathbb{R}$. *Then the following are equivalent:*

(1) $\mathbf{F}(x, y)$ *is conservative in* D.

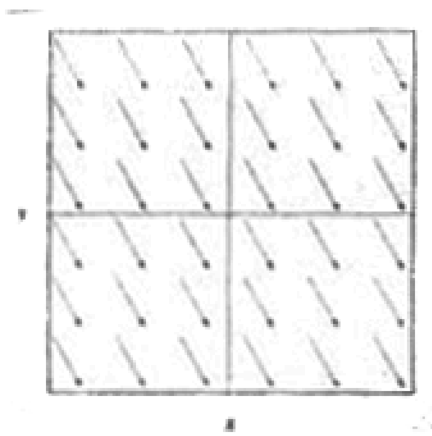
(2) $\mathbf{F}(x, y)$ *is a gradient field in* D, *i.e.,* $\mathbf{F}(x, y) = \nabla f(x, y)$ *for some potential function* f *for all* (x, y) *in* D.

(3) $\displaystyle\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ *is independent of path in* D.

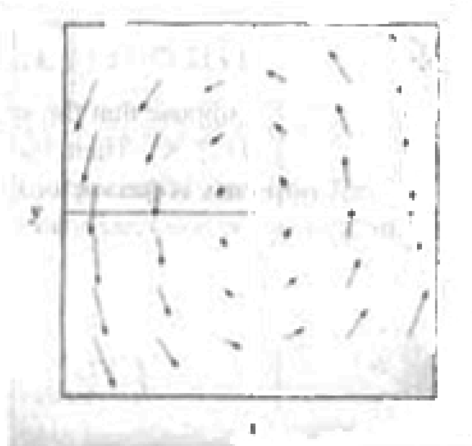
(4) $\displaystyle\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ *for every piecewise-smooth closed curve* C *lying in* D.

(5) $M_y(x, y) = N_x(x, y)$ *for all* (x, y) *in* D.

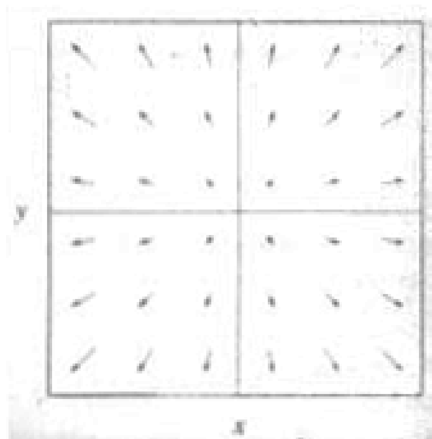
EXAMPLE. Are the following vector fields conservative?



$\mathbf{F}$ is constant $\Longrightarrow M_y = N_x = 0 \Longrightarrow$ conservative.



For C a counter-clockwise circle centered at the origin, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} > 0 \Longrightarrow$ not conservative.

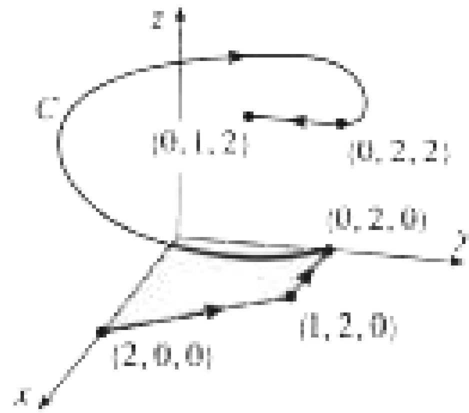


No rotation $\Longrightarrow M_y = N_x \Longrightarrow$ conservative.

Theorem (Three Dimensions). *Suppose the vector field* $\mathbf{F}(x,y,z)$ *is continuous on the open, connected region* $D \subseteq \mathbb{R}^3$. *Then* $\displaystyle\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ *is independent of path in* D *if and only if* $\mathbf{F}$ *is conservative in* D*, i.e.,* $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ *for some scalar function* f *(a potential function for* $\mathbf{F}$*) for all* (x,y,z) *in* D*. Further, for any piecewise-smooth curve* C *lying in* D *with initial point* (x_1,y_1,z_1) *and terminal point* (x_2,y_2,z_2)*,*

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = f(x,y,z)\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} = f(x_2,y_2,z_2) - f(x_1,y_1,z_1).$$

Example. Compute $\int_C \langle yz^2, xz^2, 2xyz\rangle \cdot d\mathbf{r}$ for the curve C shown below.



We need to find $f(x, y, z)$ such that

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle yz^2, xz^2, 2xyz\rangle.$$

Assume

$$f_x = yz^2 \Longrightarrow f(x, y, z) = \int yz^2\, dx = xyz^2 + g(y, z).$$

Then

$$f_y = xz^2 + g_y(y, z) \Longrightarrow g_y(y, z) = 0 \Longrightarrow g(y, z) = h(z).$$

Thus

$$f(x, y, z) = xyz^2 + h(z) \Longrightarrow f_z = 2xyz + h'(z) \Longrightarrow h'(z) = 0 \Longrightarrow h(z) = C.$$

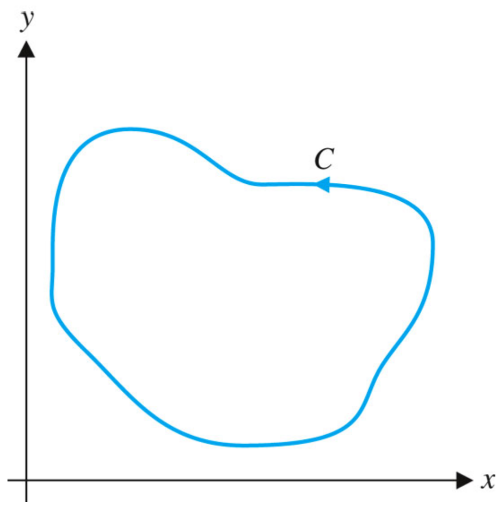
Take $C = 0$. Then $f(x, y, z) = xyz^2$ and $\nabla f = \langle yz^2, xz^2, 2xyz\rangle = \mathbf{F}$, so $\mathbf{F}$ is conservative on $\mathbb{R}^3$, and thus

$$\int_C \langle yz^2, xz^2, 2xyz\rangle \cdot d\mathbf{r} = xyz^2\Big|_{(2,0,0)}^{(0,1,2)} = 0 - 0 = 0.$$

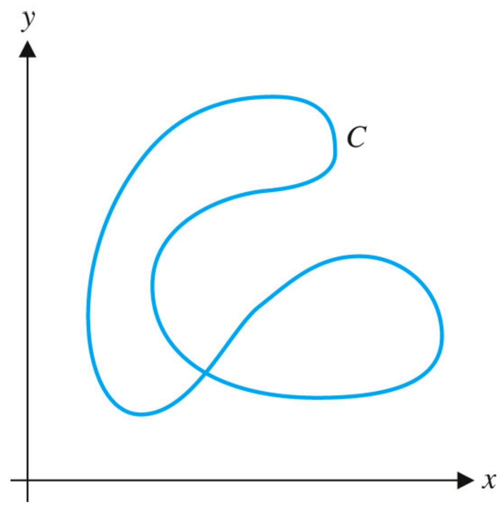
4. Green's Theorem

DEFINITION.

(1) A curve C is simple if it does not intersect itself, except at the endpoints.

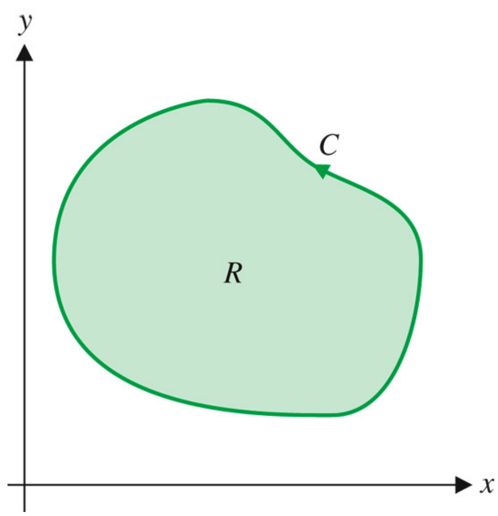


simple

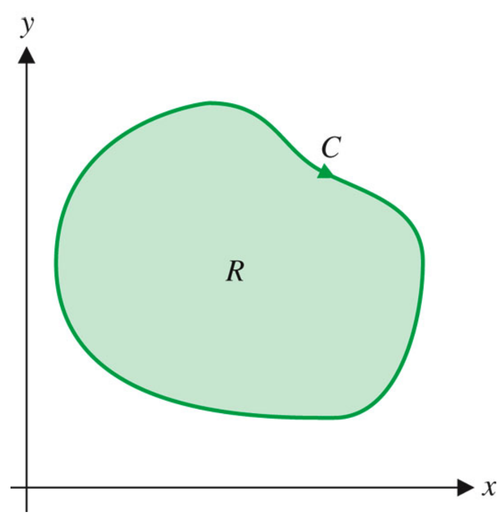


not simple

(2)A simple closed curve C has positive orientation if the region R enclosed by C stays to the left of C as the curve is traversed; a simple closed curve C has negative orientation if the region R enclosed by C stays to the right of C as the curve is traversed.



positive



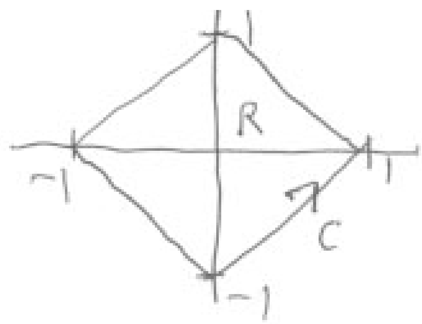
negative

NOTATION. $\oint_C F(x,y)\cdot d\mathbf{r}$ denotes a line integral along a simple closed curve C oriented in the positive direction.

THEOREM (Green's Theorem). *Let C be a piecewise-smooth simple closed curve in the plane with positive orientation and let R be the region enclosed by C. Suppose $M(x,y)$ and $N(x,y)$ are continuous and have continuous first partial derivatives in some open region D, with $R\subset D$. Then*

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

EXAMPLE. Find $\oint_C (x^4+2y)\,dx + (5x+\sin y)\,dy$.



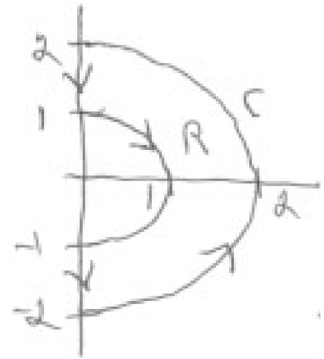
With C made up of 4 separate continuous curves (lines), direct calculation and parametrizing is cumbersome, so use Green's Theorem.

$$\oint_C (x^4+2y)\,dx + (5x+\sin y)\,dy = \iint_R (5-2)\,dA =$$

$$3\iint_R dA = 3(\text{area of } R) = 3(\sqrt{2})^2 = 3\cdot 2 = 6$$

by geometry.

EXAMPLE. Find $\oint_C (x^2y)\,dx + (x^3 + 2xy^2)\,dy$.



Again, for simplicty, use Green's Theorem.

$$\oint_C (x^2y)\,dx + (x^3 + 2xy^2)\,dy = \iint_R \left[(3x^2 + 2y^2) - x^2\right] dA =$$

$$\iint_R (2x^2 + 2y^2)\,dA = 2\iint_R (x^2 + y^2)\,dA =$$

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 r^2 r\,dr\,d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\bigg|_1^2 d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4 - \frac{1}{4}\right) d\theta =$$

$$\frac{15}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{15}{2}\theta\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{15}{2}\pi.$$

NOTATION. We use ∂R to refer to the boundary of the region R, oriented in the positive direction.

NOTE. We can replace C by ∂R in Green' Theorem.

EXAMPLE. Suppose C is a piecewise smooth, simple closed curve enclosing the region R. Then

(1) $\displaystyle\oint_C x\,dy = \iint_R (1-0)\,dA = \iint_R dA.$

(2) $\displaystyle\oint_C (-y)\,dx = \iint_R \big[0-(-1)\big]\,dA = \iint_R dA.$

Therefore,

$$\text{Area of } R = \iint_R dA = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Extending Green's Theorem

Consider R as below:

Green's Theorem doesn't apply since R is not simply connected. But we make two horizontal slits in R, dividing R into two simply-connected regions R_1 and R_2.

Apply Green's Theorem to R_1 and R_2 separately:

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA =$$
$$\oint_{\partial R_1} M(x,y)\,dx + N(x,y)\,dy + \oint_{\partial R_2} M(x,y)\,dx + N(x,y)\,dy =$$

$$\underbrace{\oint_{C_1} M(x,y)\,dx + N(x,y)\,dy + \oint_{C_2} M(x,y)\,dx + N(x,y)\,dy}_{\text{Since the line integrals over the slits cancel}} = \oint_C M(x,y)\,dx + N(x,y)\,dy$$

where $C = C_1 \cup C_2$.

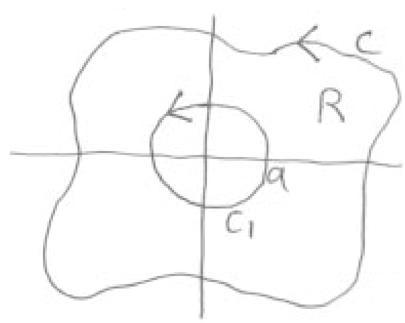
NOTE. This procedure can be extended to any finite number of holes.

EXAMPLE. Suppose $\mathbf{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$. Show that

$$\oint_C \mathbf{F}(x,y) \cdot d\mathbf{r} = 2\pi$$

for every simple closed curve C enclosing the origin.

SOLUTION. Green's Theorem doesn't apply since $\mathbf{F}(0,0)$ is not defined. Let C be any simple closed curve containing the origin and let C_1 be the circle of radius $a > 0$ centered at the origin, positively oriented, where a is sufficiently small so that C and C_1 do not meet. Let R be the region between and including the curves.



Applying the extended Green's Theorem,

$$\oint_C \mathbf{F}(x,y)\cdot d\mathbf{r} - \oint_{C_1} \mathbf{F}(x,y)\cdot d\mathbf{r} = \oint_{\partial R} \mathbf{F}(x,y)\cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA =$$

$$\iint_R \left[\frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2} - \frac{(-1)(x^2+y^2) + y(2y)}{(x^2+y^2)^2}\right] dA = \iint_R 0\, dA = 0 \Longrightarrow$$

$$\oint_C \mathbf{F}(x,y)\cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}(x,y)\cdot d\mathbf{r}.$$

Parameterize C_1 by

$$x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$$

Then $x^2 + y^2 = a^2$, and

$$\oint_C \mathbf{F}(x,y)\cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}(x,y)\cdot d\mathbf{r} = \oint_{C_1} \left\langle \frac{-y}{a^2}, \frac{x}{a^2}\right\rangle \cdot d\mathbf{r} =$$

$$\frac{1}{a^2}\oint_{C_1} (-y)\,dx + x\,dy = \frac{1}{a^2}\int_0^{2\pi} \big[(\alpha\sin t)(-a\sin t) + (a\cos t)(a\cos t)\big]\, dt =$$

$$\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = \int_0^{2\pi} dt = t\Big|_0^{2\pi} = 2\pi.$$

□