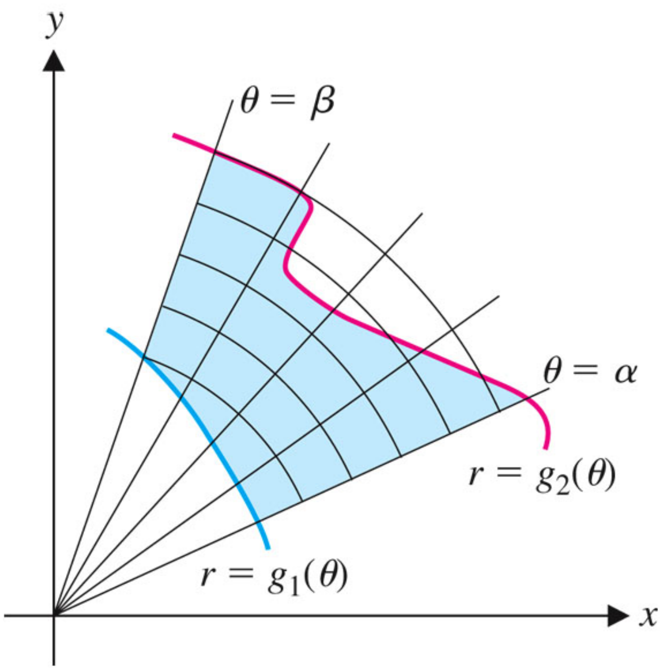
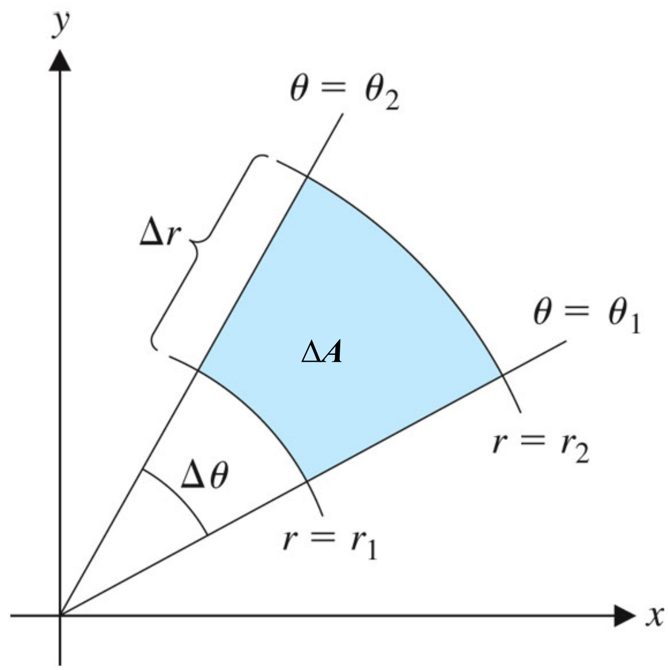
Suppose we wish to integrate $f(r,\theta)$, converted from $f(x,y)$, over a region R of the type

$$R = \{(r,\theta) | \alpha \le \theta \le \beta \text{ and } g_1(\theta) \le r \le g_2(\theta)\},$$

where $0 \le g_1(\theta) \le g_2(\theta)$ for all $\theta \in [\alpha, \beta]$, as seen below.





We make a grid of elementary polar regions (shown above on the right) and again use an inner partition.

Let $\overline{r} = \frac{1}{2}(r_1 + r_2)$, the average radius of r_1 and r_2.

$$\begin{aligned} \Delta A &= \text{area of outer sector} - \text{area of inner sector} \\ &= \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta \\ &= \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta = \overline{r}\Delta r\Delta\theta. \end{aligned}$$

Then the volume above the ith elementary polar region and below the surface is

$$V_i \approx \underbrace{f(r_i,\theta_i)}_{\text{height}} \underbrace{\Delta A_i}_{\text{area of base}} = f(r_i,\theta_i) r_i \Delta r_i \Delta\theta_i$$

where (r_i, θ_i) is a point in R_i and r_i is the average radius in R_i.