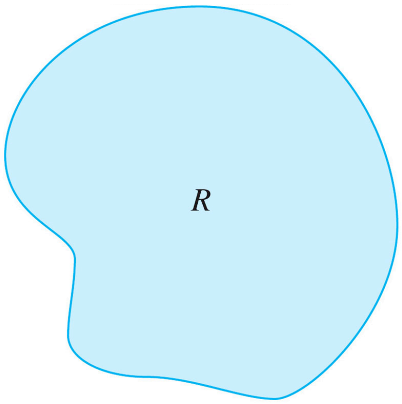
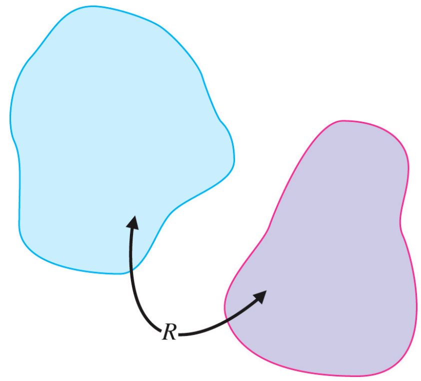
3. Independence of Path and Conservative Vector Fields

DEFINITION.

(1) A region $D \subseteq \mathbb{R}^n$ (for $n \geq 2$) is called connected if every pair of points in D can be connected by a piecewise-smooth curve lying entirely in D.



connected



not connected

(2) A path is a piecewise-smooth curve C traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$ for $a \leq t \leq b$.

(3) The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in the domain D if the integral is the same for every path contained in D that has the same beginning and end points.

THEOREM. *Suppose the vector field* $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ *is continuous on the open, connected region* $D \subseteq \mathbb{R}^2$. *Then the line integral* $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ *is independent of path if and only if* $\mathbf{F}$ *is conservative on* D.

NOTE. A similar result is valid in any number of dimensions.