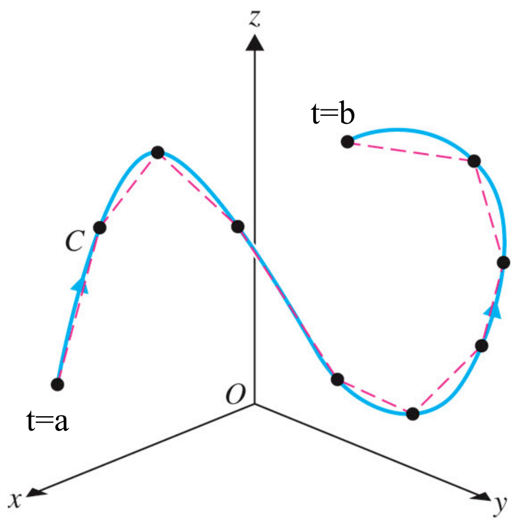
Arc Length in $\mathbb{R}^3$.

Suppose a curve

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

is traced once from $t = a$ to $t = b$.



Partition $[a, b]$ into n subintervals of equal size

$$a = t_0 < t_1 < \cdots t_n = b \text{ where } t_i - t_{i-1} = \Delta t = \frac{b-a}{n} \text{ for } i = 1, \ldots, n.$$

Let s_i = arclength from $\mathbf{r}(t_{i-1})$ to $\mathbf{r}(t_i)$.

$$\begin{aligned} s_i &\approx d(\mathbf{r}(t_{i-1}), \mathbf{r}(t_i)) \\ &= \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2} \\ &= \sqrt{[f'(c_i)\Delta t]^2 + [g'(d_i)\Delta t]^2 + [h'(e_i)\Delta t]^2} \quad \text{(by MVT)} \\ &= \sqrt{[f'(c_i)]^2 + [g'(d_i)]^2 + [h'(e_i)]^2}\,\Delta t \end{aligned}$$

For Δt small, $c_i \approx d_i \approx e_i \implies$

$$s \approx \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t \implies$$

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t$$