PROBLEM. Find $\dfrac{d}{dt}\big[f(x,y)\big]$ if x and y are functions of t.

Let $g(t) = f(x(t), y(t))$. Then

$$\begin{aligned}\frac{d}{dt}\big[f(x(t),y(t))\big] = g'(t) &= \lim_{\Delta t\to 0}\frac{g(t+\Delta t)-g(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{f\big(x(t+\Delta t),y(t+\Delta t)\big)-f\big(x(t),y(t)\big)}{\Delta t}\end{aligned}$$

$$\left\{\begin{array}{l}\text{Let } \Delta x = x(t+\Delta t)-x(t), \quad \Delta y = y(t+\Delta t)-y(t),\\ \Delta z = f\big(x(t+\Delta t),y(t+\Delta t)\big)-f\big(x(t),y(t)\big)\end{array}\right\} = \lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}$$

$$\left\{\begin{array}{l}\text{Recall } \Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y\\ \text{where } \epsilon_1,\epsilon_2 \to 0 \text{ as } (\Delta x,\Delta y)\to(0,0)\end{array}\right\}$$

$$\begin{aligned}&= \lim_{\Delta t\to 0}\frac{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y}{\Delta t}\\ &= \frac{\partial f}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} + \lim_{\Delta t\to 0}\epsilon_1\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\epsilon_2\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}.\end{aligned}$$

Now $\displaystyle\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} = \lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t} = \frac{dx}{dt}$. Similarly, $\displaystyle\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} = \frac{dy}{dt}$.

$\displaystyle\lim_{\Delta t\to 0}\Delta x = \lim_{\Delta t\to 0}\big[x(t+\Delta t)-x(t)\big] = 0$ since $x(t)$ is continuous. Similarly, $\displaystyle\lim_{\Delta t\to 0}\Delta y = 0$. Thus, since $(\Delta x,\Delta y)\to(0,0)$ as $\Delta t\to 0$, we have $\displaystyle\lim_{\Delta t\to 0}\epsilon_1 = \lim_{\Delta t\to 0}\epsilon_2 = 0$. Then

$$\begin{aligned}\frac{d}{dt}\big[f(x(t),y(t))\big] &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + 0\cdot\frac{dx}{dt} + 0\cdot\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\end{aligned}$$