EXPLORATORY EXERCISES

1. For the brachistochrone problem, two criteria for the fastest curve are: (1) steep slope at the origin and (2) concave down (note in Figure 9.16 that the positive y-axis points downward). Explain why these criteria make sense and identify other criteria. Then find parametric equations for a curve (different from the cycloid or those of exercises 13–16) that meet all the criteria. Use the formula of example 3.3 to find out how fast your curve is. You can't beat the cycloid, but get as close as you can!

2. The **tautochrone** problem is another surprising problem that was studied and solved by the same seventeenth-century mathematicians as the brachistochrone problem. (See *Journey Through Genius* by William Dunham for a description of this interesting piece of history, featuring the brilliant yet combative Bernoulli brothers.) Recall that the cycloid of example 3.3 runs from (0, 0) to $(\pi, 2)$. It takes the skier $k\sqrt{2}\pi = \pi/g$ seconds to ski the path. How long would it take the skier starting partway down the path, for instance, at $(\pi/2 - 1, 1)$? Find the slope of the cycloid at this point and compare it to the slope at (0, 0). Explain why the skier would build up less speed starting at this new point. Graph the speed function for the cycloid with $0 \le u \le 1$ and explain why the farther down the slope you start, the less speed you'll have. To see how speed and distance balance, use the time formula

$$T = \frac{\pi}{g}\int_a^1 \frac{\sqrt{1-\cos \pi u}}{\sqrt{\cos \pi a - \cos \pi u}}\, du$$

for the time it takes to ski the cycloid starting at the point $(\pi a - \sin \pi a, 1 - \cos \pi a)$, $0 < a < 1$. What is the remarkable property that the cycloid has?

9.4 POLAR COORDINATES

You've probably heard the cliche about how difficult it is to try to fit a round peg into a square hole. In some sense, we have faced this problem on several occasions so far in our study of calculus. For instance, if we were to use an integral to calculate the area of the circle $x^2 + y^2 = 9$, we would have

$$A = \int_{-3}^{3}\left[\sqrt{9-x^2} - \left(-\sqrt{9-x^2}\right)\right] dx = 2\int_{-3}^{3}\sqrt{9-x^2}\, dx. \tag{4.1}$$

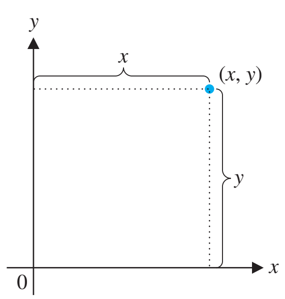


FIGURE 9.21
Rectangular coordinates

Note that you can evaluate this integral by making the trigonometric substitution $x = 3\sin\theta$. (It's a good thing that we already know a simple formula for the area of a circle!) A better plan might be to use parametric equations, such as $x = 3\cos t$, $y = 3\sin t$, for $0 \le t \le 2\pi$, to describe the circle. In section 9.2, we saw that the area is given by

$$\int_0^{2\pi} x(t)y'(t)\, dt = \int_0^{2\pi} (3\cos t)(3\cos t)\, dt = 9\int_0^{2\pi} \cos^2 t\, dt.$$

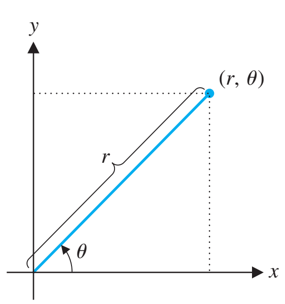


FIGURE 9.22
Polar coordinates

This is certainly better than the integral in (4.1), but it still requires some effort to evaluate this. The basic problem is that circles do not translate well into the usual x-y coordinate system. We often refer to this system as a system of **rectangular coordinates,** because a point is described in terms of the horizontal and vertical distances from the origin (see Figure 9.21).

An alternative description of a point in the xy-plane consists of specifying the distance r from the point to the origin and an angle θ (in radians) measured from the positive x-axis counterclockwise to the ray connecting the point and the origin (see Figure 9.22). We describe the point by the ordered pair (r, θ) and refer to r and θ as **polar coordinates** for the point.

EXAMPLE 4.1 Converting from Polar to Rectangular Coordinates

Plot the points with the indicated polar coordinates and determine the corresponding rectangular coordinates (x, y) for: (a) $(2, 0)$, (b) $(3, \frac{\pi}{2})$, (c) $(-3, \frac{\pi}{2})$ and (d) $(2, \pi)$.

Solution (a) Notice that the angle $\theta = 0$ locates the point on the positive x-axis. At a distance of $r = 2$ units from the origin, this corresponds to the point $(2, 0)$ in rectangular coordinates (see Figure 9.23a).

(b) The angle $\theta = \frac{\pi}{2}$ locates points on the positive y-axis. At a distance of $r = 3$ units from the origin, this corresponds to the point $(0, 3)$ in rectangular coordinates (see Figure 9.23b).

(c) The angle is the same as in (b), but a negative value of r indicates that the point is located 3 units in the opposite direction, at the point $(0, -3)$ in rectangular coordinates (see Figure 9.23b).

(d) The angle $\theta = \pi$ corresponds to the negative x-axis. The distance of $r = 2$ units from the origin gives us the point $(-2, 0)$ in rectangular coordinates (see Figure 9.23c).

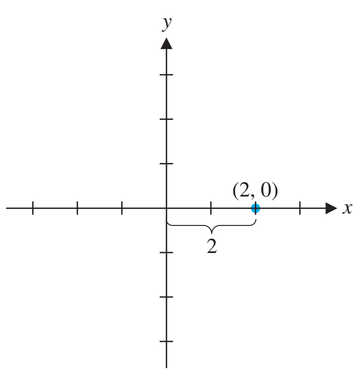


FIGURE 9.23a
The point $(2, 0)$ in polar coordinates

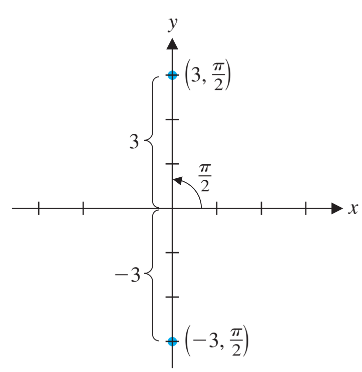


FIGURE 9.23b
The points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ in polar coordinates

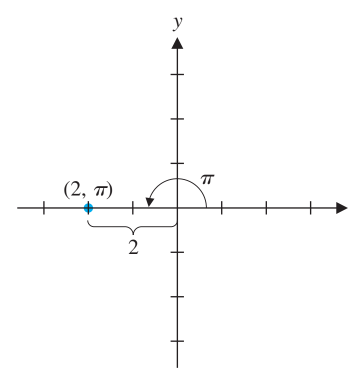


FIGURE 9.23c
The point $(2, \pi)$ in polar coordinates

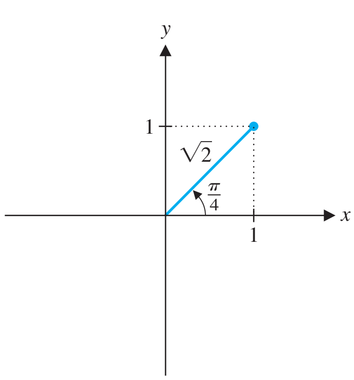


FIGURE 9.24a
Polar coordinates for the point $(1, 1)$

EXAMPLE 4.2 Converting from Rectangular to Polar Coordinates

Find a polar coordinate representation of the rectangular point $(1, 1)$.

Solution From Figure 9.24a, notice that the point lies on the line $y = x$, which makes an angle of $\frac{\pi}{4}$ with the positive x-axis. From the distance formula, we get that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. This says that we can write the point as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates. Referring to Figure 9.24b (on the following page), notice that we can specify the same point by using a negative value of r, $r = -\sqrt{2}$, with the angle $\frac{5\pi}{4}$. (Think about this some.) Notice further, that the angle $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ corresponds to the

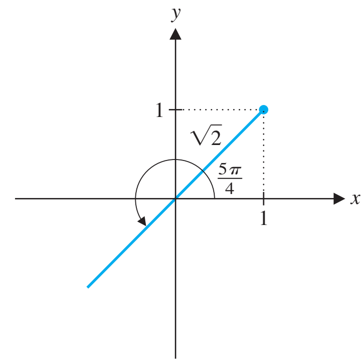


FIGURE 9.24b
An alternative polar representation of (1, 1)

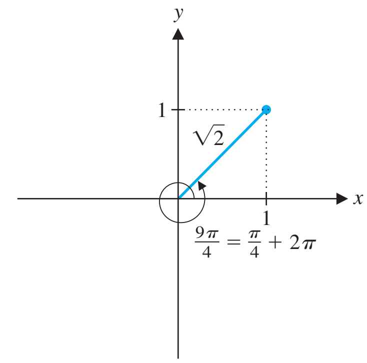


FIGURE 9.24c
Another polar representation of the point (1, 1)

same ray shown in Figure 9.24a (see Figure 9.24c). In fact, all of the polar points $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ and $(-\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ for any integer n correspond to the same point in the xy-plane. ■

REMARK 4.1

As we see in example 4.2, each point (x, y) in the plane has infinitely many polar coordinate representations. For a given angle θ, the angles $\theta \pm 2\pi$, $\theta \pm 4\pi$ and so on, all correspond to the same ray. For convenience, we use the notation $\theta + 2n\pi$ (for any integer n) to represent all of these possible angles.

Referring to Figure 9.25, notice that it is a simple matter to find the rectangular coordinates (x, y) of a point specified in polar coordinates as (r, θ). From the usual definitions for $\sin\theta$ and $\cos\theta$, we get

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{4.2}$$

From equations (4.2), notice that for a point (x, y) in the plane,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and for $x \neq 0$,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

That is, every polar coordinate representation (r, θ) of the point (x, y), where $x \neq 0$ must satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}. \tag{4.3}$$

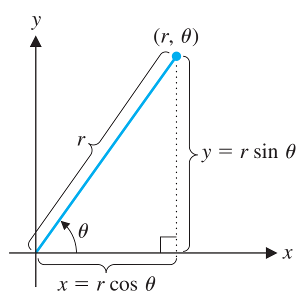


FIGURE 9.25
Converting from polar to rectangular coordinates

Notice that since there's more than one choice of r and θ, we cannot actually solve equations (4.3) to produce formulas for r and θ. In particular, while you might be tempted to write $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, this is not the only possible choice. Remember that for (r, θ) to be a polar representation of the point (x, y), θ can be *any* angle for which $\tan\theta = \frac{y}{x}$, while $\tan^{-1}\left(\frac{y}{x}\right)$ gives you an angle θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Finding polar coordinates for a given point is typically a process involving some graphing and some thought.

EXAMPLE 4.3 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations for the rectangular points (a) (2, 3) and (b) (−3, 1).

Solution (a) With $x = 2$ and $y = 3$, we have from (4.3) that

$$r^2 = x^2 + y^2 = 2^2 + 3^2 = 13,$$

REMARK 4.2

Notice that for any point (x, y) specified in rectangular coordinates $(x \neq 0)$, we can always write the point in polar coordinates using either of the polar angles $\tan^{-1}\left(\frac{y}{x}\right)$ or $\tan^{-1}\left(\frac{y}{x}\right) + \pi$. You can determine which angle corresponds to $r = \sqrt{x^2 + y^2}$ and which corresponds to $r = -\sqrt{x^2 + y^2}$ by looking at the quadrant in which the point lies.

so that $r = \pm\sqrt{13}$. Also,

$$\tan\theta = \frac{y}{x} = \frac{3}{2}.$$

One angle is then $\theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 0.98$ radian. To determine which choice of r corresponds to this angle, note that (2, 3) is located in the first quadrant (see Figure 9.26a). Since 0.98 radian also puts you in the first quadrant, this angle corresponds to the positive value of r, so that $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)\right)$ is one polar representation of the point. The negative choice of r corresponds to an angle one half-circle (i.e., π radians) away (see Figure 9.26b), so that another representation is $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + \pi\right)$. Every other polar representation is found by adding multiples of 2π to the two angles used above. That is, every polar representation of the point (2, 3) must have the form $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + 2n\pi\right)$ or $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + \pi + 2n\pi\right)$, for some integer choice of n.

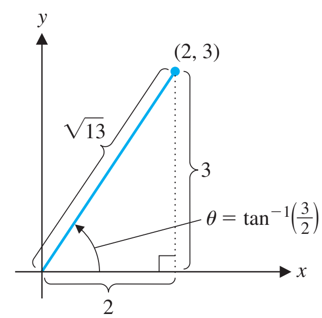


FIGURE 9.26a
The point (2, 3)

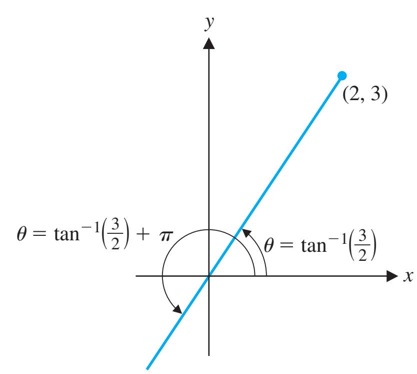


FIGURE 9.26b
Negative value of r

(b) For the point (−3, 1), we have $x = -3$ and $y = 1$. From (4.3), we have

$$r^2 = x^2 + y^2 = (-3)^2 + 1^2 = 10,$$

so that $r = \pm\sqrt{10}$. Further,

$$\tan\theta = \frac{y}{x} = \frac{1}{-3},$$

so that the most obvious choice for the polar angle is $\theta = \tan^{-1}\left(-\frac{1}{3}\right) \approx -0.32$, which lies in the fourth quadrant. Since the point (−3, 1) is in the second quadrant, this choice of the angle corresponds to the negative value of r (see Figure 9.27). The positive value of r then corresponds to the angle $\theta = \tan^{-1}\left(-\frac{1}{3}\right) + \pi$. Observe that all polar coordinate representations must then be of the form $\left(-\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right) + 2n\pi\right)$ or $\left(\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right) + \pi + 2n\pi\right)$, for some integer choice of n. ■

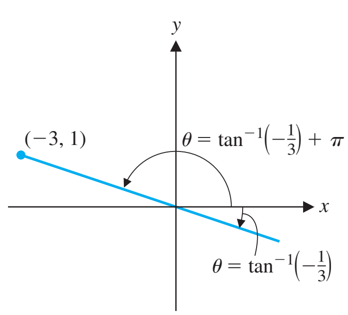


FIGURE 9.27
The point (−3, 1)

Observe that the conversion from polar coordinates to rectangular coordinates is completely straightforward, as in example 4.4.

EXAMPLE 4.4 Converting from Polar to Rectangular Coordinates

Find the rectangular coordinates for the polar points (a) $\left(3, \frac{\pi}{6}\right)$ and (b) (−2, 3).

Solution For (a), we have from (4.2) that

$$x = r\cos\theta = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

and $$y = r\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

The rectangular point is then $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. For (b), we have

$$x = r\cos\theta = -2\cos 3 \approx 1.98$$

and $$y = r\sin\theta = -2\sin 3 \approx -0.28.$$

The rectangular point is $(-2\cos 3, -2\sin 3)$, which is located at approximately $(1.98, -0.28)$. ■

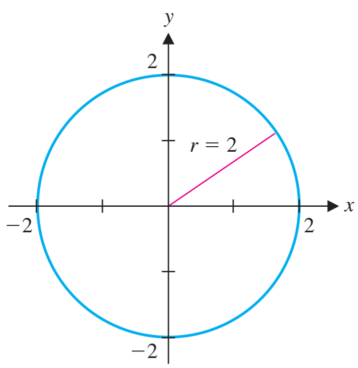


FIGURE 9.28a
The circle $r = 2$

The **graph** of a polar equation $r = f(\theta)$ is the set of all points (x, y) for which $x = r\cos\theta$, $y = r\sin\theta$ and $r = f(\theta)$. In other words, the graph of a polar equation is a graph in the xy-plane of all those points whose polar coordinates satisfy the given equation. We begin by sketching two very simple (and familiar) graphs. The key to drawing the graph of a polar equation is to always keep in mind what the polar coordinates represent.

EXAMPLE 4.5 Some Simple Graphs in Polar Coordinates

Sketch the graphs of (a) $r = 2$ and (b) $\theta = \pi/3$.

Solution For (a), notice that $2 = r = \sqrt{x^2 + y^2}$ and so, we want all points whose distance from the origin is 2 (with *any* polar angle θ). Of course, this is the definition of a circle of radius 2 with center at the origin (see Figure 9.28a). For (b), notice that $\theta = \pi/3$ specifies all points with a polar angle of $\pi/3$ from the positive x-axis (at *any* distance r from the origin). Including negative values for r, this defines a line with slope $\tan\pi/3 = \sqrt{3}$ (see Figure 9.28b). ■

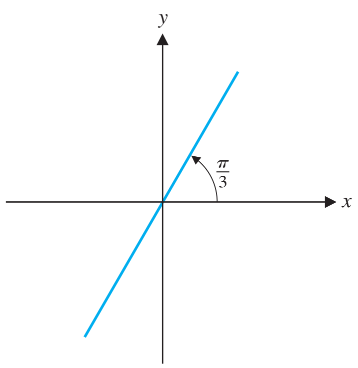


FIGURE 9.28b
The line $\theta = \frac{\pi}{3}$

It turns out that many familiar curves have simple polar equations.

EXAMPLE 4.6 Converting an Equation from Rectangular to Polar Coordinates

Find the polar equation(s) corresponding to the hyperbola $x^2 - y^2 = 9$ (see Figure 9.29).

Solution From (4.2), we have

$$\begin{aligned} 9 = x^2 - y^2 &= r^2\cos^2\theta - r^2\sin^2\theta \\ &= r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta. \end{aligned}$$

Solving for r, we get

$$r^2 = \frac{9}{\cos 2\theta} = 9\sec 2\theta,$$

so that $$r = \pm 3\sqrt{\sec 2\theta}.$$

Notice that in order to keep $\sec 2\theta > 0$, we can restrict 2θ to lie in the interval $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$, so that $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Observe that with this range of values of θ, the hyperbola is drawn exactly once, where $r = 3\sqrt{\sec 2\theta}$ corresponds to the right branch of the hyperbola and $r = -3\sqrt{\sec 2\theta}$ corresponds to the left branch. ■

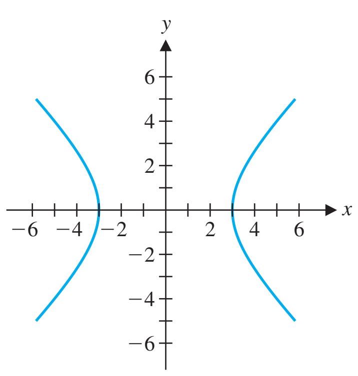


FIGURE 9.29
$x^2 - y^2 = 9$

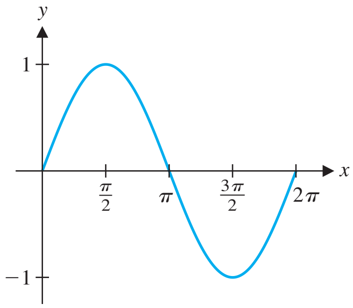


FIGURE 9.30a
$y = \sin x$ plotted in *rectangular* coordinates

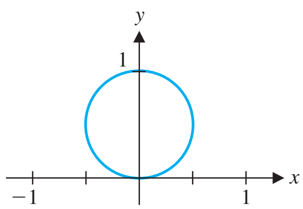


FIGURE 9.30b
The circle $r = \sin\theta$

EXAMPLE 4.7 A Surprisingly Simple Polar Graph

Sketch the graph of the polar equation $r = \sin\theta$.

Solution For reference, we first sketch a graph of the sine function in *rectangular* coordinates on the interval $[0, 2\pi]$ (see Figure 9.30a). Notice that on the interval $0 \le \theta \le \frac{\pi}{2}$, $\sin\theta$ increases from 0 to its maximum value of 1. This corresponds to a polar arc in the first quadrant from the origin ($r = 0$) to 1 unit up on the y-axis. Then, on the interval $\frac{\pi}{2} \le \theta \le \pi$, $\sin\theta$ decreases from 1 to 0. This corresponds to an arc in the second quadrant, from 1 unit up on the y-axis back to the origin. Next, on the interval $\pi \le \theta \le \frac{3\pi}{2}$, $\sin\theta$ decreases from 0 to its minimum value of -1. Since the values of r are negative, remember that this means that the points plotted are in the *opposite* quadrant (i.e., the first quadrant). Notice that this traces out the same curve in the first quadrant as we've already drawn for $0 \le \theta \le \frac{\pi}{2}$. Likewise, taking θ in the interval $\frac{3\pi}{2} \le \theta \le 2\pi$ retraces the portion of the curve in the second quadrant. Since $\sin\theta$ is periodic of period 2π, taking further values of θ simply retraces portions of the curve that we have already drawn. A sketch of the polar graph is shown in Figure 9.30b. We now verify that this curve is actually a circle. Notice that if we multiply the equation $r = \sin\theta$ through by r, we get

$$r^2 = r\sin\theta.$$

You should immediately recognize from (4.2) and (4.3) that $y = r\sin\theta$ and $r^2 = x^2 + y^2$. This gives us the rectangular equation

$$x^2 + y^2 = y$$

or

$$0 = x^2 + y^2 - y.$$

Completing the square, we get

$$0 = x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$

or, adding $\frac{1}{4}$ to both sides,

$$\left(\frac{1}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\right)^2.$$

This is the rectangular equation for the circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$, which is what we see in Figure 9.30b. ■

The graphs of many polar equations are not the graphs of *any* functions of the form $y = f(x)$, as in example 4.8.

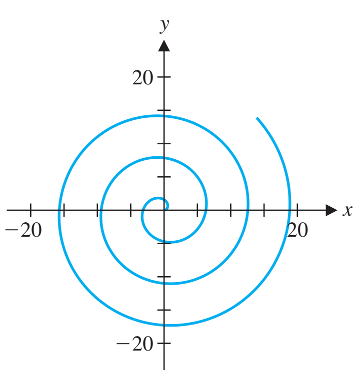


FIGURE 9.31
The spiral $r = \theta, \theta \ge 0$

EXAMPLE 4.8 An Archimedian Spiral

Sketch the graph of the polar equation $r = \theta$, for $\theta \ge 0$.

Solution Notice that here, as θ increases, so too does r. That is, as the polar angle increases, the distance from the origin also increases accordingly. This produces the spiral (an example of an **Archimedian spiral**) seen in Figure 9.31. ■

The graphs shown in examples 4.9, 4.10 and 4.11 are all in the general class known as **limaçons.** This class of graphs is defined by $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$, for positive constants a and b. If $a = b$, the graphs are called **cardioids.**

EXAMPLE 4.9 A Limaçon

Sketch the graph of the polar equation $r = 3 + 2\cos\theta$.

Solution We begin by sketching the graph of $y = 3 + 2\cos x$ in rectangular coordinates on the interval $[0, 2\pi]$, to use as a reference (see Figure 9.32). Notice that in this case, we have $r = 3 + 2\cos\theta > 0$ for all values of θ. Further, the maximum value of r is 5 (corresponding to when $\cos\theta = 1$ at $\theta = 0, 2\pi$, etc.) and the minimum value of r is 1 (corresponding to when $\cos\theta = -1$ at $\theta = \pi, 3\pi$, etc.). In this case, the polar graph is traced out with $0 \le \theta \le 2\pi$. We summarize the intervals of increase and decrease for r in the following table.

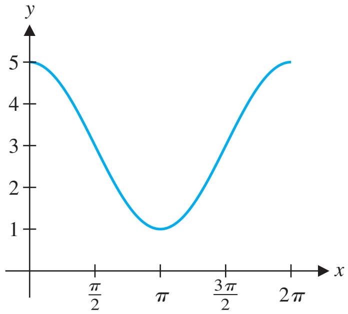


FIGURE 9.32
$y = 3 + 2\cos x$ in rectangular coordinates

Interval	$\cos\theta$	$r = 3 + 2\cos\theta$
$\left[0, \frac{\pi}{2}\right]$	Decreases from 1 to 0	Decreases from 5 to 3
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 0 to -1	Decreases from 3 to 1
$\left[\pi, \frac{3\pi}{2}\right]$	Increases from -1 to 0	Increases from 1 to 3
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from 0 to 1	Increases from 3 to 5

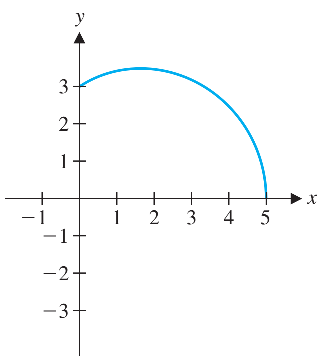


FIGURE 9.33a
$0 \le \theta \le \frac{\pi}{2}$

In Figures 9.33a–9.33d, we show how the sketch progresses through each interval indicated in the table, with the completed figure (called a **limaçon**) shown in Figure 9.33d.

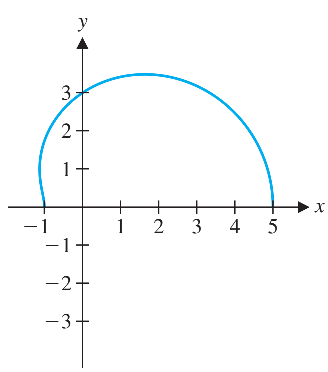


FIGURE 9.33b
$0 \le \theta \le \pi$

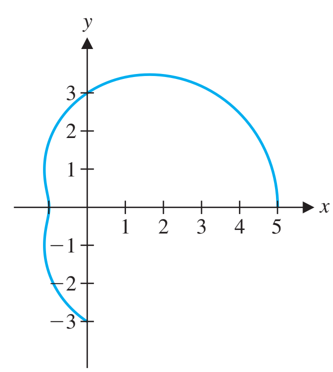


FIGURE 9.33c
$0 \le \theta \le \frac{3\pi}{2}$

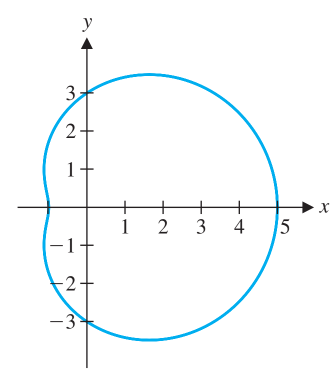


FIGURE 9.33d
$0 \le \theta \le 2\pi$

EXAMPLE 4.10 The Graph of a Cardioid

Sketch the graph of the polar equation $r = 2 - 2\sin\theta$.

Solution As we have done several times now, we first sketch a graph of $y = 2 - 2\sin x$ in rectangular coordinates, on the interval $[0, 2\pi]$, as in Figure 9.34. We summarize the intervals of increase and decrease in the following table.

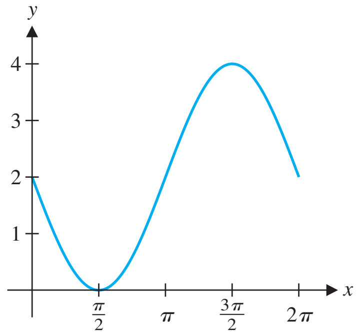


FIGURE 9.34
$y = 2 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 2 - 2\sin\theta$
$\left[0, \frac{\pi}{2}\right]$	Increases from 0 to 1	Decreases from 2 to 0
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 1 to 0	Increases from 0 to 2
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to -1	Increases from 2 to 4
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from -1 to 0	Decreases from 4 to 2

Again, we sketch the graph in stages, corresponding to each of the intervals indicated in the table, as seen in Figures 9.35a–9.35d.

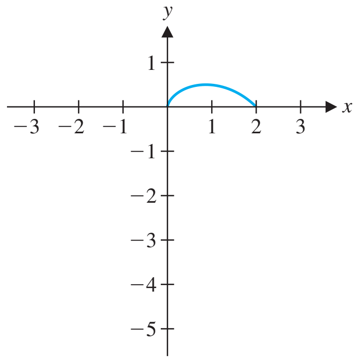


FIGURE 9.35a
$0 \le \theta \le \frac{\pi}{2}$

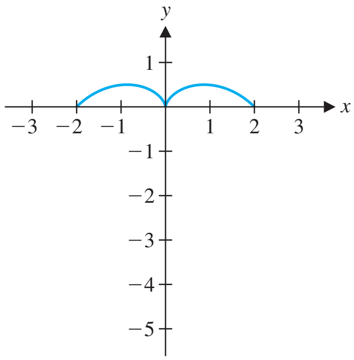


FIGURE 9.35b
$0 \le \theta \le \pi$

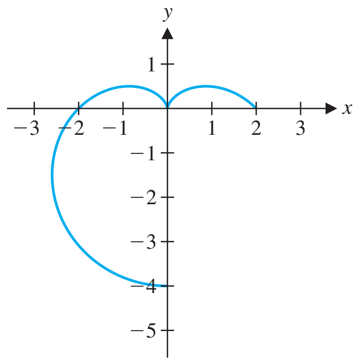


FIGURE 9.35c
$0 \le \theta \le \frac{3\pi}{2}$

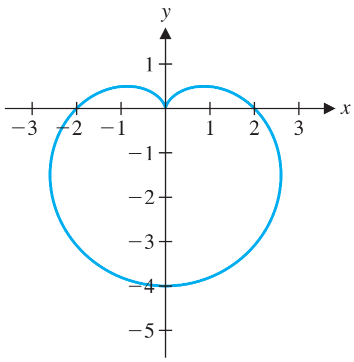


FIGURE 9.35d
$0 \le \theta \le 2\pi$

The completed graph appears in Figure 9.35d and is sketched out for $0 \le \theta \le 2\pi$. You can see why this figure is called a **cardioid** ("heartlike"). ■

EXAMPLE 4.11 A Limaçon with a Loop

Sketch the graph of the polar equation $r = 1 - 2\sin\theta$.

Solution We again begin by sketching a graph of $y = 1 - 2\sin x$ in rectangular coordinates, as in Figure 9.36. We summarize the intervals of increase and decrease in the following table.

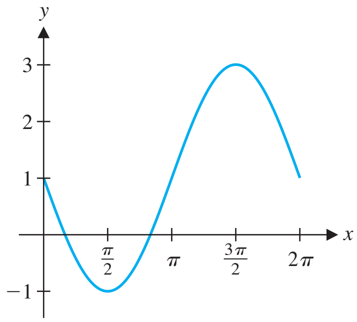


FIGURE 9.36
$y = 1 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$
$[0, \frac{\pi}{2}]$	Increases from 0 to 1	Decreases from 1 to -1
$[\frac{\pi}{2}, \pi]$	Decreases from 1 to 0	Increases from -1 to 1
$[\pi, \frac{3\pi}{2}]$	Decreases from 0 to -1	Increases from 1 to 3
$[\frac{3\pi}{2}, 2\pi]$	Increases from -1 to 0	Decreases from 3 to 1

Notice that since r assumes both positive and negative values in this case, we need to exercise a bit more caution, as negative values for r cause us to draw that portion of the graph in the *opposite* quadrant. Observe that $r = 0$ when $1 - 2\sin\theta = 0$, that is, when $\sin\theta = \frac{1}{2}$. This will occur when $\theta = \frac{\pi}{6}$ and when $\theta = \frac{5\pi}{6}$. For this reason, we expand the above table, to include more intervals and where we also indicate the quadrant where the graph is to be drawn, as follows:

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$	*Quadrant*
$[0, \frac{\pi}{6}]$	Increases from 0 to $\frac{1}{2}$	Decreases from 1 to 0	First
$[\frac{\pi}{6}, \frac{\pi}{2}]$	Increases from $\frac{1}{2}$ to 1	Decreases from 0 to -1	Third
$[\frac{\pi}{2}, \frac{5\pi}{6}]$	Decreases from 1 to $\frac{1}{2}$	Increases from -1 to 0	Fourth
$[\frac{5\pi}{6}, \pi]$	Decreases from $\frac{1}{2}$ to 0	Increases from 0 to 1	Second
$[\pi, \frac{3\pi}{2}]$	Decreases from 0 to -1	Increases from 1 to 3	Third
$[\frac{3\pi}{2}, 2\pi]$	Increases from -1 to 0	Decreases from 3 to 1	Fourth

We sketch the graph in stages in Figures 9.37a–9.37f, corresponding to each of the intervals indicated in the table.

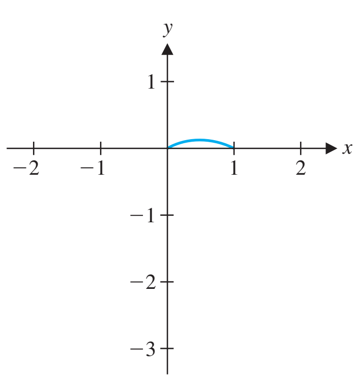


FIGURE 9.37a
$0 \le \theta \le \frac{\pi}{6}$

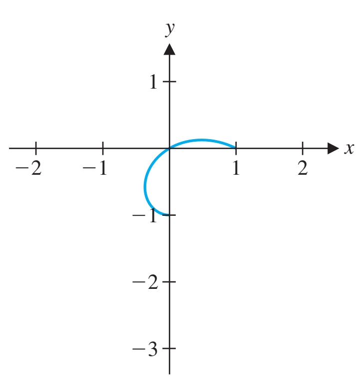


FIGURE 9.37b
$0 \le \theta \le \frac{\pi}{2}$

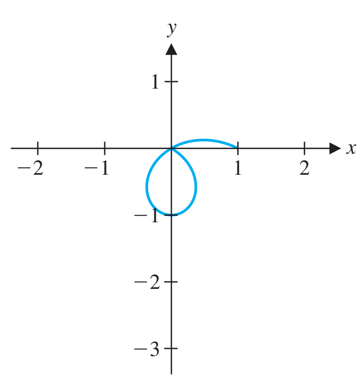


FIGURE 9.37c
$0 \le \theta \le \frac{5\pi}{6}$

The completed graph appears in Figure 9.37f and is sketched out for $0 \le \theta \le 2\pi$. You should observe from this the importance of determining where $r = 0$, as well as where r is increasing and decreasing.

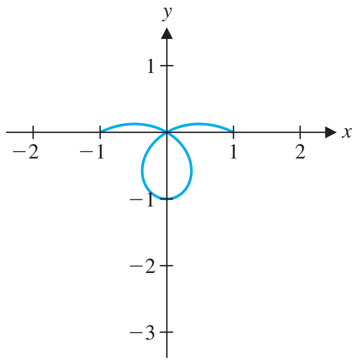


FIGURE 9.37d
$0 \le \theta \le \pi$

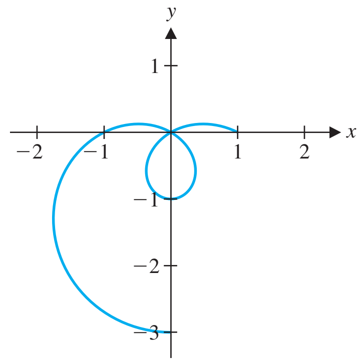


FIGURE 9.37e
$0 \le \theta \le \frac{3\pi}{2}$

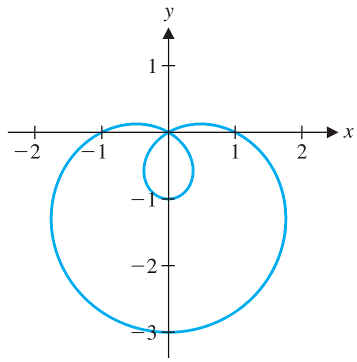


FIGURE 9.37f
$0 \le \theta \le 2\pi$

EXAMPLE 4.12 A Four-Leaf Rose

Sketch the graph of the polar equation $r = \sin 2\theta$.

Solution As usual, we will first draw a graph of $y = \sin 2x$ in rectangular coordinates on the interval $[0, 2\pi]$, as seen in Figure 9.38. Notice that the period of $\sin 2\theta$ is only π. We summarize the intervals on which the function is increasing and decreasing in the following table.

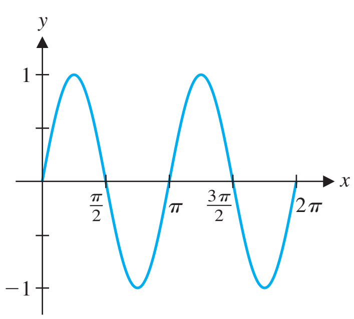


FIGURE 9.38
$y = \sin 2x$ in rectangular coordinates

Interval	$r = \sin 2\theta$	*Quadrant*
$\left[0, \frac{\pi}{4}\right]$	Increases from 0 to 1	First
$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$	Decreases from 1 to 0	First
$\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$	Decreases from 0 to -1	Fourth
$\left[\frac{3\pi}{4}, \pi\right]$	Increases from -1 to 0	Fourth
$\left[\pi, \frac{5\pi}{4}\right]$	Increases from 0 to 1	Third
$\left[\frac{5\pi}{4}, \frac{3\pi}{2}\right]$	Decreases from 1 to 0	Third
$\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right]$	Decreases from 0 to -1	Second
$\left[\frac{7\pi}{4}, 2\pi\right]$	Increases from -1 to 0	Second

We sketch the graph in stages in Figures 9.39a–9.39h, each one corresponding to the intervals indicated in the table, where we have also indicated the lines $y = \pm x$, as a guide.

This is an interesting curve known as a **four-leaf rose.** Notice again the significance of the points corresponding to $r = 0$, or $\sin 2\theta = 0$. Also, notice that r reaches a maximum of 1 when $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ or $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$ and r reaches a minimum of -1 when $2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ or $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$. Again, you must keep in mind that when the value of r is negative, this causes us to draw the graph in the opposite quadrant.

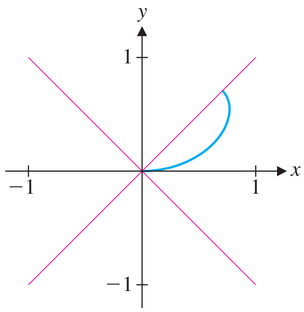


FIGURE 9.39a
$0 \le \theta \le \frac{\pi}{4}$

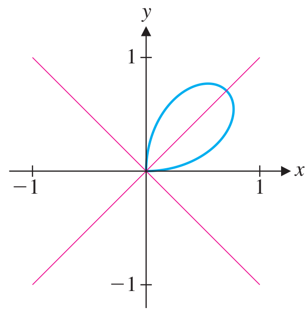


FIGURE 9.39b
$0 \le \theta \le \frac{\pi}{2}$

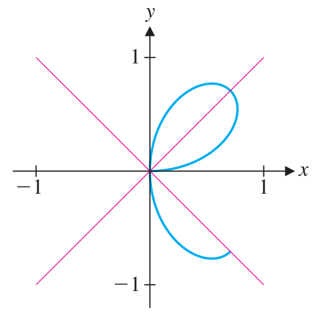


FIGURE 9.39c
$0 \le \theta \le \frac{3\pi}{4}$

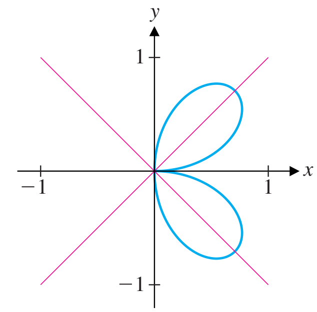


FIGURE 9.39d
$0 \le \theta \le \pi$

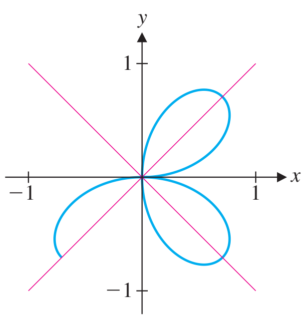


FIGURE 9.39e
$0 \le \theta \le \frac{5\pi}{4}$

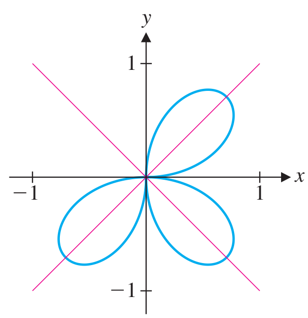


FIGURE 9.39f
$0 \le \theta \le \frac{3\pi}{2}$

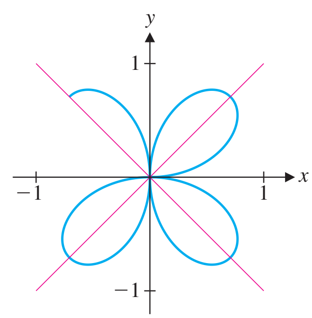


FIGURE 9.39g
$0 \le \theta \le \frac{7\pi}{4}$

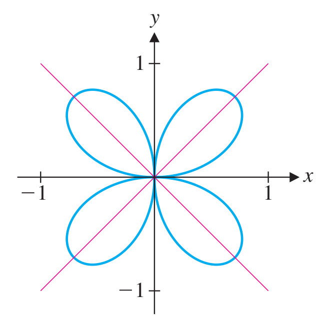


FIGURE 9.39h
$0 \le \theta \le 2\pi$

Note that in example 4.12, even though the period of the function $\sin 2\theta$ is π, it took θ-values ranging from 0 to 2π to sketch the entire curve $r = \sin 2\theta$. By contrast, the period of the function $\sin\theta$ is 2π, but the circle $r = \sin\theta$ was completed with $0 \le \theta \le \pi$. To determine the range of values of θ that produces a graph, you need to carefully identify important points as we did in example 4.12. The Trace feature found on graphing calculators can be very helpful for getting an idea of the θ-range, but remember that such Trace values are only approximate.

You will explore a variety of other interesting graphs in the exercises.

BEYOND FORMULAS

The graphics in Figures 9.35, 9.37 and 9.39 provide a good visual model of how to think of polar graphs. Most polar graphs $r = f(\theta)$ can be sketched as a sequence of connected arcs, where the arcs start and stop at places where $r = 0$ or where a new quadrant is entered. By breaking the larger graph into small arcs, you can use the properties of f to quickly determine where each arc starts and stops.

EXERCISES 9.4

WRITING EXERCISES

1. Suppose a point has polar representation (r, θ). Explain why another polar representation of the same point is $(-r, \theta + \pi)$.
2. After working with rectangular coordinates for so long, the idea of polar representations may seem slightly awkward. However, polar representations are entirely natural in many settings. For instance, if you were on a ship at sea and another ship was approaching you, explain whether you would use a polar representation (distance and bearing) or a rectangular representation (distance east-west and distance north-south).
3. In example 4.7, the graph (a circle) of $r = \sin\theta$ is completely traced out with $0 \le \theta \le \pi$. Explain why graphing $r = \sin\theta$ with $\pi \le \theta \le 2\pi$ would produce the same full circle.
4. Two possible advantages of introducing a new coordinate system are making previous problems easier to solve and allowing new problems to be solved. Give two examples of graphs for which the polar equation is simpler than the rectangular equation. Give two examples of polar graphs for which you have not seen a rectangular equation.

In exercises 1–6, plot the given polar points (r, θ) and find their rectangular representation.

1. $(2, 0)$ 2. $(2, \pi)$ 3. $(-2, \pi)$
4. $\left(-3, \frac{3\pi}{2}\right)$ 5. $(3, -\pi)$ 6. $\left(5, -\frac{\pi}{2}\right)$

In exercises 7–12, find all polar coordinate representations of the given rectangular point.

7. $(2, -2)$ 8. $(-1, 1)$ 9. $(0, 3)$
10. $(2, -1)$ 11. $(3, 4)$ 12. $(-2, -\sqrt{5})$

In exercises 13–18, find rectangular coordinates for the given polar point.

13. $\left(2, -\frac{\pi}{3}\right)$ 14. $\left(-1, \frac{\pi}{3}\right)$ 15. $(0, 3)$
16. $\left(3, \frac{\pi}{8}\right)$ 17. $\left(4, \frac{\pi}{10}\right)$ 18. $(-3, 1)$

In exercises 19–26, sketch the graph of the polar equation and find a corresponding x-y equation.

19. $r = 4$ 20. $r = \sqrt{3}$ 21. $\theta = \pi/6$
22. $\theta = 3\pi/4$ 23. $r = \cos\theta$ 24. $r = 2\cos\theta$
25. $r = 3\sin\theta$ 26. $r = 2\sin\theta$

In exercises 27–50, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

27. $r = \cos 2\theta$ 28. $r = \cos 3\theta$
29. $r = \sin 3\theta$ 30. $r = \sin 2\theta$
31. $r = 3 + 2\sin\theta$ 32. $r = 2 - 2\cos\theta$
33. $r = 2 - 4\sin\theta$ 34. $r = 2 + 4\cos\theta$
35. $r = 2 + 2\sin\theta$ 36. $r = 3 - 6\cos\theta$
37. $r = \frac{1}{4}\theta$ 38. $r = e^{\theta/4}$
39. $r = 2\cos(\theta - \pi/4)$ 40. $r = 2\sin(3\theta - \pi)$
41. $r = \cos\theta + \sin\theta$ 42. $r = \cos\theta + \sin 2\theta$
43. $r = \tan^{-1} 2\theta$ 44. $r = \theta/\sqrt{\theta^2 + 1}$
45. $r = 2 + 4\cos 3\theta$ 46. $r = 2 - 4\sin 4\theta$
47. $r = \dfrac{2}{1 + \sin\theta}$ 48. $r = \dfrac{3}{1 - \sin\theta}$
49. $r = \dfrac{2}{1 + \cos\theta}$ 50. $r = \dfrac{3}{1 - \cos\theta}$

51. Graph $r = 4\cos\theta\sin^2\theta$ and explain why there is no curve to the left of the y-axis.
52. Graph $r = \theta\cos\theta$ for $-2\pi \le \theta \le 2\pi$. Explain why this is called the Garfield curve.

53. Based on your graphs in exercises 23 and 24, conjecture the graph of $r = a\cos\theta$ for any positive constant a.
54. Based on your graphs in exercises 25 and 26, conjecture the graph of $r = a\sin\theta$ for any positive constant a.
55. Based on the graphs in exercises 27 and 28 and others (try $r = \cos 4\theta$ and $r = \cos 5\theta$), conjecture the graph of $r = \cos n\theta$ for any positive integer n.
56. Based on the graphs in exercises 29 and 30 and others (try $r = \sin 4\theta$ and $r = \sin 5\theta$), conjecture the graph of $r = \sin n\theta$ for any positive integer n.

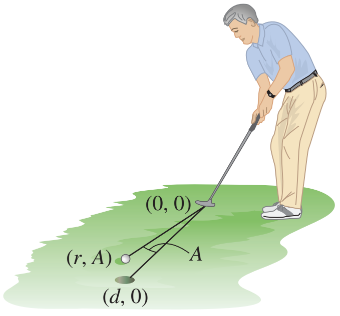
In exercises 57–62, find a polar equation corresponding to the given rectangular equation.

57. $y^2 - x^2 = 4$ 58. $x^2 + y^2 = 9$
59. $x^2 + y^2 = 16$ 60. $x^2 + y^2 = x$
61. $y = 3$ 62. $x = 2$

63. Sketch the graph of $r = \cos \frac{11}{12}\theta$ first for $0 \le \theta \le \pi$, then for $0 \le \theta \le 2\pi$, then for $0 \le \theta \le 3\pi, \ldots$, and finally for $0 \le \theta \le 24\pi$. Discuss any patterns that you find and predict what will happen for larger domains.

64. Sketch the graph of $r = \cos \pi\theta$ first for $0 \le \theta \le 1$, then for $0 \le \theta \le 2$, then for $0 \le \theta \le 3, \ldots$ and finally for $0 \le \theta \le 20$. Discuss any patterns that you find and predict what will happen for larger domains.

65. One situation where polar coordinates apply directly to sports is in making a golf putt. The two factors that the golfer tries to control are distance (determined by speed) and direction (usually called the "line"). Suppose a putter is d feet from the hole, which has radius $h = \frac{1}{6}'$. Show that the path of the ball will intersect the hole if the angle A in the figure satisfies $-\sin^{-1}(h/d) < A < \sin^{-1}(h/d)$.



66. The distance r that the golf ball in exercise 65 travels also needs to be controlled. The ball must reach the front of the hole. In rectangular coordinates, the hole has equation $(x - d)^2 + y^2 = h^2$, so the left side of the hole is $x = d - \sqrt{h^2 - y^2}$. Show that this converts in polar coordinates to $r = d\cos\theta - \sqrt{d^2\cos^2\theta - (d^2 - h^2)}$. (Hint: Substitute for x and y, isolate the square root term, square both sides, combine r^2 terms and use the quadratic formula.)

67. The golf putt in exercises 65 and 66 will not go in the hole if it is hit too hard. Suppose that the putt would go $r = d + c$ feet if it did not go in the hole ($c > 0$). For a putt hit toward the center of the hole, define b to be the largest value of c such that the putt goes in (i.e., if the ball is hit more than b feet past the hole, it is hit too hard). Experimental evidence (see Dave Pelz's *Putt Like the Pros*) shows that at other angles A, the distance r must be less than $d + b\left(1 - \left[\dfrac{A}{\sin^{-1}(h/d)}\right]^2\right)$. The results of exercises 65 and 66 define limits for the angle A and distance r of a successful putt. Identify the functions $r_1(A)$ and $r_2(A)$ such that $r_1(A) < r < r_2(A)$ and constants A_1 and A_2 such that $A_1 < A < A_2$.

68. Take the general result of exercise 67 and apply it to a putt of $d = 15$ feet with a value of $b = 4$ feet. Visualize this by graphing the region

$$15\cos\theta - \sqrt{225\cos^2\theta - (225 - 1/36)} < r < 15 + 4\left(1 - \left[\frac{\theta}{\sin^{-1}(1/90)}\right]^2\right)$$

with $-\sin^{-1}(1/90) < \theta < \sin^{-1}(1/90)$. A good choice of graphing windows is $13.8 \le x \le 19$ and $-0.5 \le y \le 0.5$.

EXPLORATORY EXERCISES

1. In this exercise, you will explore the roles of the constants a, b and c in the graph of $r = af(b\theta + c)$. To start, sketch $r = \sin\theta$ followed by $r = 2\sin\theta$ and $r = 3\sin\theta$. What does the constant a affect? Then sketch $r = \sin(\theta + \pi/2)$ and $r = \sin(\theta - \pi/4)$. What does the constant c affect? Now for the tough one. Sketch $r = \sin 2\theta$ and $r = \sin 3\theta$. What does the constant b seem to affect? Test all of your hypotheses on the base function $r = 1 + 2\cos\theta$ and several functions of your choice.

2. The polar curve $r = ae^{b\theta}$ is sometimes called an **equiangular curve.** To see why, sketch the curve and then show that $\dfrac{dr}{d\theta} = br$. A somewhat complicated geometric argument shows that $\dfrac{dr}{d\theta} = r\cot\alpha$, where α is the angle between the tangent line and the line connecting the point on the curve to the origin. Comparing equations, conclude that the angle α is constant (hence "equiangular"). To illustrate this property, compute α for the points at $\theta = 0$ and $\theta = \pi$ for $r = e^{\theta}$. This type of spiral shows up often in nature, possibly because the equal-angle property can be easily achieved. Spirals can be found among shellfish (the picture shown here is of an ammonite fossil from about 350 million years ago) and the florets of the common daisy. Other examples, including the connection to sunflowers, the Fibonacci sequence and the musical scale, can be found in H. E. Huntley's *The Divine Proportion*.

9.5 CALCULUS AND POLAR COORDINATES

Having introduced polar coordinates and looked at a variety of polar graphs, our next step is to extend the techniques of calculus to the case of polar coordinates. In this section, we focus on tangent lines, area and arc length. Surface area and other applications will be examined in the exercises.

Notice that you can think of the graph of the polar equation $r = f(\theta)$ as the graph of the parametric equations $x = f(t)\cos t$, $y(t) = f(t)\sin t$ (where we have used the parameter $t = \theta$), since from (4.2)

$$x = r\cos\theta = f(\theta)\cos\theta \tag{5.1}$$

and

$$y = r\sin\theta = f(\theta)\sin\theta. \tag{5.2}$$

In view of this, we can now take any results already derived for parametric equations and extend these to the special case of polar coordinates.

In section 9.2, we showed that the slope of the tangent line at the point corresponding to $\theta = a$ is given [from (2.1)] to be

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{\dfrac{dy}{d\theta}(a)}{\dfrac{dx}{d\theta}(a)}.} \tag{5.3}$$

From the product rule, (5.1) and (5.2), we have

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

and

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta.$$

Putting these together with (5.3), we get

$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{f'(a)\sin a + f(a)\cos a}{f'(a)\cos a - f(a)\sin a}. \tag{5.4}$$

EXAMPLE 5.1 Finding the Slope of the Tangent Line to a Three-Leaf Rose

Find the slope of the tangent line to the three-leaf rose $r = \sin 3\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{4}$.

Solution A sketch of the curve is shown in Figure 9.40a. From (4.2), we have

$$y = r\sin\theta = \sin 3\theta \sin\theta$$

and

$$x = r\cos\theta = \sin 3\theta\cos\theta.$$

Using (5.3), we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{(3\cos 3\theta)\sin\theta + \sin 3\theta(\cos\theta)}{(3\cos 3\theta)\cos\theta - \sin 3\theta(\sin\theta)}. \tag{5.5}$$

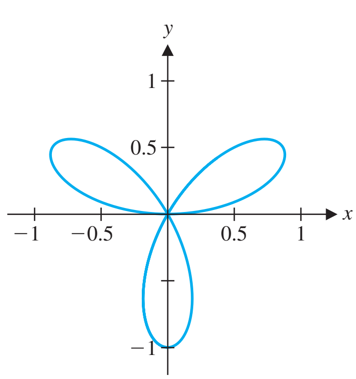


FIGURE 9.40a
Three-leaf rose

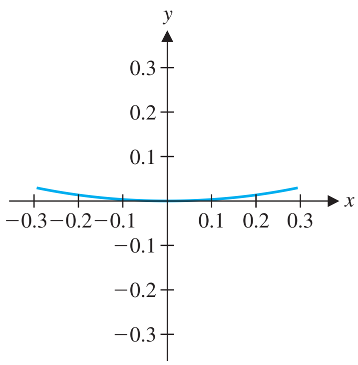


FIGURE 9.40b
$-0.1 \le \theta \le 0.1$

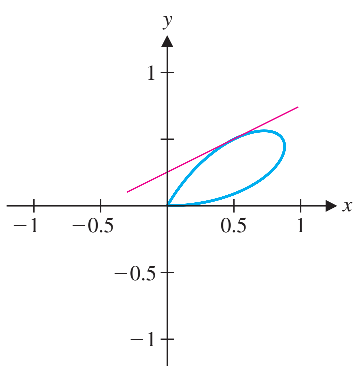


FIGURE 9.40c
The tangent line at $\theta = \frac{\pi}{4}$

At $\theta = 0$, this gives us

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{(3\cos 0)\sin 0 + \sin 0(\cos 0)}{(3\cos 0)\cos 0 - \sin 0(\sin 0)} = \frac{0}{3} = 0.$$

In Figure 9.40b, we sketch $r = \sin 3\theta$ for $-0.1 \le \theta \le 0.1$, in order to isolate the portion of the curve around $\theta = 0$. Notice that from this figure, a slope of 0 seems reasonable.

Similarly, at $\theta = \frac{\pi}{4}$, we have from (5.5) that

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\left(3\cos\frac{3\pi}{4}\right)\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\left(\cos\frac{\pi}{4}\right)}{\left(3\cos\frac{3\pi}{4}\right)\cos\frac{\pi}{4} - \sin\frac{3\pi}{4}\left(\sin\frac{\pi}{4}\right)} = \frac{-\frac{3}{2}+\frac{1}{2}}{-\frac{3}{2}-\frac{1}{2}} = \frac{1}{2}.$$

In Figure 9.40c, we show the section of $r = \sin 3\theta$ for $0 \le \theta \le \frac{\pi}{3}$, along with the tangent line at $\theta = \frac{\pi}{4}$. ■

Recall that for functions $y = f(x)$, horizontal tangents were especially significant for locating maximum and minimum points. For polar graphs, the significant points are often places where r has reached a maximum or minimum, which may or may not correspond to a horizontal tangent. We explore this idea further in example 5.2.

EXAMPLE 5.2 Polar Graphs and Horizontal Tangent Lines

For the three-leaf rose $r = \sin 3\theta$, find the locations of all horizontal tangent lines and interpret the significance of these points. Further, at the three points where $|r|$ is a maximum, show that the tangent line is perpendicular to the line segment connecting the point to the origin.

Solution From (5.3) and (5.4), we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Here, $f(\theta) = \sin 3\theta$ and so, to have $\frac{dy}{dx} = 0$, we must have

$$0 = \frac{dy}{d\theta} = 3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta.$$

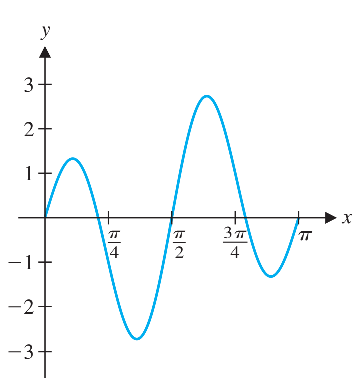


FIGURE 9.41a
$y = 3\cos 3x \sin x + \sin 3x \cos x$

Solving this equation is not an easy matter. As a start, we graph $f(x) = 3\cos 3x \sin x + \sin 3x \cos x$ with $0 \le x \le \pi$ (see Figure 9.41a). You should observe that there appear to be five solutions. Three of the solutions can be found exactly: $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$. You can find the remaining two numerically: $\theta \approx 0.659$ and $\theta \approx 2.48$. (You can also use trig identities to arrive at $\sin^2\theta = \frac{3}{8}$.) The corresponding points on the curve $r = \sin 3\theta$ (specified in rectangular coordinates) are $(0, 0)$, $(0.73, 0.56)$, $(0, -1)$, $(-0.73, 0.56)$ and $(0, 0)$. The point $(0, -1)$ lies at the bottom of a leaf. This is the familiar situation of a horizontal tangent line at a local (and in fact, absolute) minimum. The point $(0, 0)$ is a little more tricky to interpret. As seen in Figure 9.40b, if we graph a small piece of the curve with θ near 0 (or π), the point $(0, 0)$ is a minimum point. However, this is not true for other values of θ (e.g., $\frac{\pi}{3}$) where the curve passes through the point $(0, 0)$. The tangent lines at the points $(\pm 0.73, 0.56)$ are shown in Figure 9.41b. Note that these points correspond to points where the y-coordinate is a maximum. However, referring to the graph, these points do not appear to be of particular

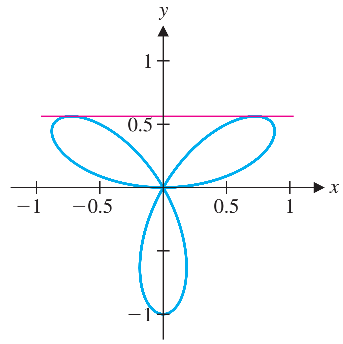


FIGURE 9.41b
Horizontal tangent lines

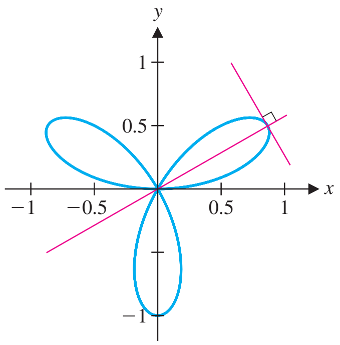


FIGURE 9.41c
The tangent line at the tip of a leaf

interest. Rather, the tips of the leaves represent the extreme points of most interest. Notice that the tips are where $|r|$ is a maximum. For $r = \sin 3\theta$, this occurs when $\sin 3\theta = \pm 1$, that is, where $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, or $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$. From (5.4), the slope of the tangent line to the curve at $\theta = \frac{\pi}{6}$ is given by

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{\left(3\cos\frac{3\pi}{6}\right)\sin\frac{\pi}{6} + \sin\frac{3\pi}{6}\left(\cos\frac{\pi}{6}\right)}{\left(3\cos\frac{3\pi}{6}\right)\cos\frac{\pi}{6} - \sin\frac{3\pi}{6}\left(\sin\frac{\pi}{6}\right)} = \frac{0 + \frac{\sqrt{3}}{2}}{0 - \frac{1}{2}} = -\sqrt{3}.$$

The rectangular point corresponding to $\theta = \frac{\pi}{6}$ is given by

$$\left(1\cos\frac{\pi}{6}, 1\sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

The slope of the line segment joining this point to the origin is then $\frac{1}{\sqrt{3}}$. Observe that the line segment from the origin to the point is perpendicular to the tangent line since the product of the slopes ($-\sqrt{3}$ and $\frac{1}{\sqrt{3}}$) is -1. This is illustrated in Figure 9.41c. Similarly, the slope of the tangent line at $\theta = \frac{5\pi}{6}$ is $\sqrt{3}$, which again makes the tangent line at that point perpendicular to the line segment from the origin to the point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Finally, we have already shown that the slope of the tangent line at $\theta = \frac{\pi}{2}$ is 0 and a horizontal tangent line is perpendicular to the vertical line from the origin to the point $(0, -1)$. ■

Next, for polar curves like the three-leaf rose seen in Figure 9.40a, we would like to compute the area enclosed by the curve. Since such a graph is *not* the graph of a function of the form $y = f(x)$, we cannot use the usual area formulas developed in Chapter 5. While we can convert our area formulas for parametric equations (from Theorem 2.2) into polar coordinates, a simpler approach uses the following geometric argument.

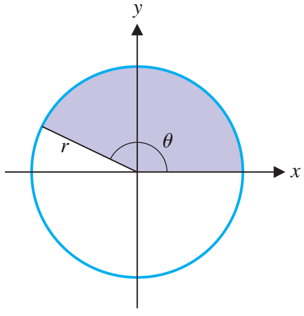


FIGURE 9.42
Circular sector

Observe that a sector of a circle of radius r and central angle θ, measured in radians (see Figure 9.42) contains a fraction $\left(\frac{\theta}{2\pi}\right)$ of the area of the entire circle. So, the area of the sector is given by

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta.$$

Now, consider the area enclosed by the polar curve defined by the equation $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ (see Figure 9.43a), where f is continuous and positive on the interval $a \leq \theta \leq b$. As we did when we defined the definite integral, we begin by partitioning the

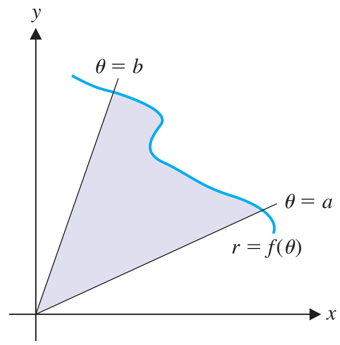


FIGURE 9.43a
Area of a polar region

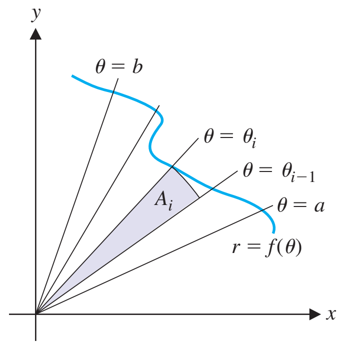


FIGURE 9.43b
Approximating the area of a polar region

θ-interval into n equal pieces:

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = b.$$

The width of each of these subintervals is then $\Delta\theta = \theta_i - \theta_{i-1} = \dfrac{b-a}{n}$. (Does this look familiar?) On each subinterval $[\theta_{i-1}, \theta_i]$ $(i = 1, 2, \ldots, n)$, we approximate the curve with the circular arc $r = f(\theta_i)$ (see Figure 9.43b). The area A_i enclosed by the curve on this subinterval is then approximately the same as the area of the circular sector of radius $f(\theta_i)$ and central angle $\Delta\theta$:

$$A_i \approx \frac{1}{2}r^2\Delta\theta = \frac{1}{2}[f(\theta_i)]^2\Delta\theta.$$

The total area A enclosed by the curve is then approximately the same as the sum of the areas of all such circular sectors:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\Delta\theta.$$

As we have done numerous times now, we can improve the approximation by making n larger. Taking the limit as $n \to \infty$ gives us a definite integral:

Area in polar coordinates

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta. \tag{5.6}$$

EXAMPLE 5.3 The Area of One Leaf of a Three-Leaf Rose

Find the area of one leaf of the rose $r = \sin 3\theta$.

Solution Notice that one leaf of the rose is traced out with $0 \le \theta \le \frac{\pi}{3}$ (see Figure 9.44). From (5.6), the area is given by

$$A = \int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta\,d\theta$$
$$= \frac{1}{4}\int_0^{\pi/3}(1 - \cos 6\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/3} = \frac{\pi}{12},$$

where we have used the half-angle formula $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ to simplify the integrand. ■

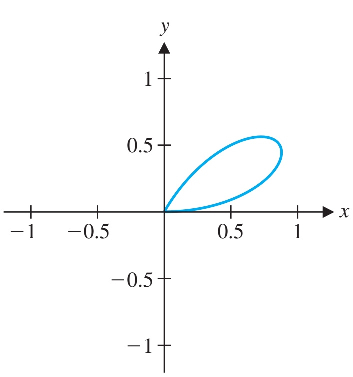


FIGURE 9.44
One leaf of $r = \sin 3\theta$

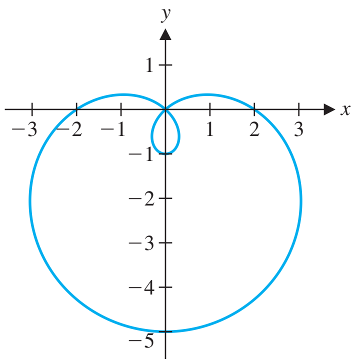


FIGURE 9.45
$r = 2 - 3\sin\theta$

Often, the most challenging part of finding the area of a polar region is determining the limits of integration.

EXAMPLE 5.4 The Area of the Inner Loop of a Limaçon

Find the area of the inner loop of the limaçon $r = 2 - 3\sin\theta$.

Solution A sketch of the limaçon is shown in Figure 9.45. Starting at $\theta = 0$, the curve starts at the point (2, 0), passes through the origin, traces out the inner loop, passes back through the origin and finally traces out the outer loop. Thus, the inner loop is formed by θ-values between the first and second occurrences of $r = 0$ with $\theta > 0$. Solving $r = 0$, we get $\sin\theta = \frac{2}{3}$. The two smallest positive solutions are $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ and $\theta = \pi - \sin^{-1}\left(\frac{2}{3}\right)$. Numerically, these are approximately equal to $\theta = 0.73$ and $\theta = 2.41$. From (5.6), the area is approximately

$$A \approx \int_{0.73}^{2.41} \frac{1}{2}(2 - 3\sin\theta)^2\, d\theta = \frac{1}{2}\int_{0.73}^{2.41} (4 - 12\sin\theta + 9\sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{0.73}^{2.41} \left[4 - 12\sin\theta + \frac{9}{2}(1 - \cos 2\theta)\right] d\theta \approx 0.44,$$

where we have used the half-angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to simplify the integrand. (Here the area is approximate, owing only to the approximate limits of integration.) ■

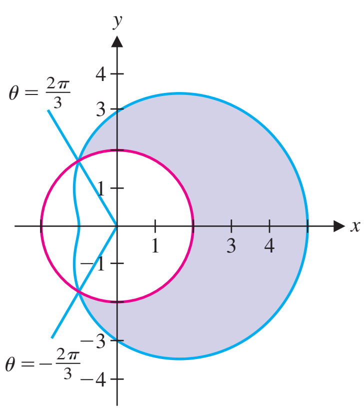


FIGURE 9.46a
$r = 3 + 2\cos\theta$ and $r = 2$

When finding the area lying between two polar graphs, we use the familiar device of subtracting one area from another. Although the calculations in example 5.5 aren't too messy, finding the points of intersection of two polar curves often provides the greatest challenge.

EXAMPLE 5.5 Finding the Area between Two Polar Graphs

Find the area inside the limaçon $r = 3 + 2\cos\theta$ and outside the circle $r = 2$.

Solution We show a sketch of the two curves in Figure 9.46a. Notice that the limits of integration correspond to the values of θ where the two curves intersect. So, we must first solve the equation $3 + 2\cos\theta = 2$. Notice that since $\cos\theta$ is periodic, there are infinitely many solutions of this equation. Consequently, it is essential to consult the graph to determine which solutions you are interested in. In this case, we want the least negative and the smallest positive solutions. (Look carefully at Figure 9.46b, where we have shaded the area between the graphs corresponding to θ between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, the first two positive solutions. This portion of the graphs corresponds to the area *outside* the limaçon and *inside* the circle!) With $3 + 2\cos\theta = 2$, we have $\cos\theta = -\frac{1}{2}$, which occurs at $\theta = -\frac{2\pi}{3}$ and $\theta = \frac{2\pi}{3}$. From (5.6), the area enclosed by the portion of the limaçon on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\, d\theta = \frac{33\sqrt{3} + 44\pi}{6}.$$

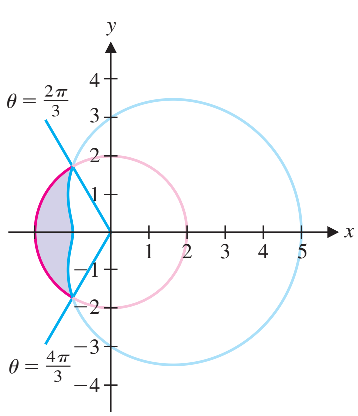


FIGURE 9.46b
$\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$

Similarly, the area enclosed by the circle on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\, d\theta = \frac{8\pi}{3}.$$

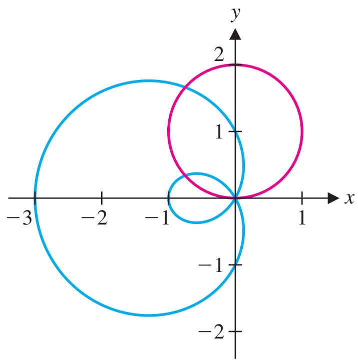


FIGURE 9.47a
$r = 1 - 2\cos\theta$ and $r = 2\sin\theta$

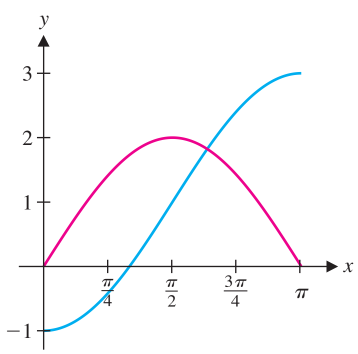


FIGURE 9.47b
Rectangular plot:
$y = 1 - 2\cos x$, $y = 2\sin x$,
$0 \le x \le \pi$

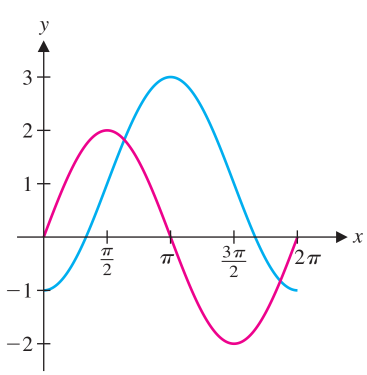


FIGURE 9.47c
Rectangular plot: $y = 1 - 2\cos x$,
$y = 2\sin x$, $0 \le x \le 2\pi$

The area *inside* the limaçon and *outside* the circle is then given by

$$A = \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\, d\theta - \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\, d\theta$$
$$= \frac{33\sqrt{3} + 44\pi}{6} - \frac{8\pi}{3} = \frac{33\sqrt{3} + 28\pi}{6} \approx 24.2.$$

Here, we have left the (routine) details of the integrations to you. ■

In cases where r takes on both positive and negative values, finding the intersection points of two curves is more complicated.

EXAMPLE 5.6 Finding Intersections of Polar Curves Where r Can Be Negative

Find all intersections of the limaçon $r = 1 - 2\cos\theta$ and the circle $r = 2\sin\theta$.

Solution We show a sketch of the two curves in Figure 9.47a. Notice from the sketch that there are three intersections of the two curves. Since $r = 2\sin\theta$ is completely traced with $0 \le \theta \le \pi$, you might reasonably expect to find three solutions of the equation $1 - 2\cos\theta = 2\sin\theta$ on the interval $0 \le \theta \le \pi$. However, if we draw a rectangular plot of the two curves $y = 1 - 2\cos x$ and $y = 2\sin x$, on the interval $0 \le x \le \pi$ (see Figure 9.47b), we can clearly see that there is only one solution in this range, at approximately $\theta \approx 1.99$. (Use Newton's method or your calculator's solver to obtain an accurate approximation.) The corresponding rectangular point is $(r\cos\theta, r\sin\theta) \approx (-0.74, 1.67)$. Looking at Figure 9.47a, observe that there is another intersection located below this point. One way to find this point is to look at a rectangular plot of the two curves corresponding to an expanded range of values of θ (see Figure 9.47c). Notice that there is a second solution of the equation $1 - 2\cos\theta = 2\sin\theta$, near $\theta = 5.86$, which corresponds to the point $(-0.74, 0.34)$. Note that this point is on the inner loop of $r = 1 - 2\cos\theta$ and corresponds to a negative value of r. Finally, there appears to be a third intersection at or near the origin. Notice that this does not arise from any solution of the equation $1 - 2\cos\theta = 2\sin\theta$. This is because, while both curves pass through the origin (You should verify this!), they each do so for *different* values of θ. (Keep in mind that the origin corresponds to the point $(0, \theta)$, in polar coordinates, for *any* angle θ.) Notice that $1 - 2\cos\theta = 0$ for $\theta = \frac{\pi}{3}$ and $2\sin\theta = 0$ for $\theta = 0$. So, although the curves intersect at the origin, they each pass through the origin for different values of θ. ■

REMARK 5.1

To find points of intersection of two polar curves $r = f(\theta)$ and $r = g(\theta)$, you must keep in mind that points have more than one representation in polar coordinates. In particular, this says that points of intersection need not correspond to solutions of $f(\theta) = g(\theta)$.

In example 5.7, we see an application that is far simpler to set up in polar coordinates than in rectangular coordinates.

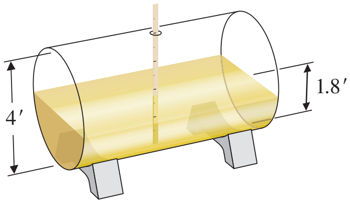


FIGURE 9.48a
A cylindrical oil tank

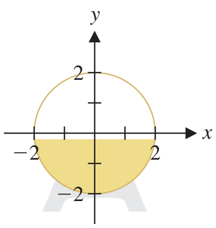


FIGURE 9.48b
Cross section of tank

EXAMPLE 5.7 Finding the Volume of a Partially Filled Cylinder

A cylindrical oil tank with a radius of 2 feet is lying on its side. A measuring stick shows that the oil is 1.8 feet deep (see Figure 9.48a). What percentage of a full tank is left?

Solution Notice that since we wish to find the *percentage* of oil remaining in the tank, the length of the tank has no bearing on this problem. (Think about this some.) We need only consider a cross section of the tank, which we represent as a circle of radius 2 centered at the origin. The proportion of oil remaining is given by the area of that portion of the circle lying beneath the line $y = -0.2$, divided by the total area of the circle. The area of the circle is 4π, so we need only find the area of the shaded region in Figure 9.48b. Computing this area in rectangular coordinates is a mess (try it!), but it is straightforward in polar coordinates. First, notice that the line $y = -0.2$ corresponds to $r \sin\theta = -0.2$ or $r = -0.2 \csc\theta$. The area beneath the line and inside the circle is then given by (5.6) as

$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\, d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2\csc\theta)^2\, d\theta,$$

where θ_1 and θ_2 are the appropriate intersections of $r = 2$ and $r = -0.2 \csc\theta$. Using Newton's method, the first two positive solutions of $2 = -0.2 \csc\theta$ are $\theta_1 \approx 3.242$ and $\theta_2 \approx 6.183$. The area is then

$$\begin{aligned}\text{Area} &= \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\, d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2\csc\theta)^2\, d\theta \\ &= (2\theta + 0.02\cot\theta)\Big|_{\theta_1}^{\theta_2} \approx 5.485.\end{aligned}$$

The fraction of oil remaining in the tank is then approximately $5.485/4\pi \approx 0.43648$ or about 43.6% of the total capacity of the tank. ■

We close this section with a brief discussion of arc length for polar curves. Recall that from (3.1), the arc length of a curve defined parametrically by $x = x(t)$, $y = y(t)$, for $a \le t \le b$, is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{5.7}$$

Once again thinking of a polar curve as a parametric representation (where the parameter is θ), we have that for the polar curve $r = f(\theta)$,

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

This gives us

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2 \\ &= [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + f'(\theta)f(\theta)(-2\cos\theta\sin\theta + 2\sin\theta\cos\theta) \\ &\quad + [f(\theta)]^2(\cos^2\theta + \sin^2\theta) \\ &= [f'(\theta)]^2 + [f(\theta)]^2.\end{aligned}$$

From (5.7), the arc length is then

Arc length in polar coordinates

$$s = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta. \tag{5.8}$$

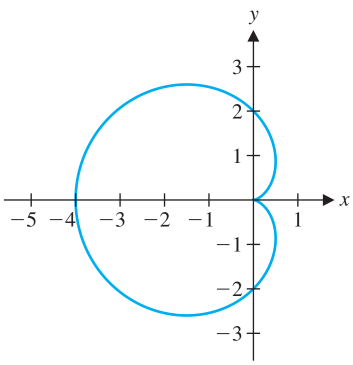


FIGURE 9.49
$r = 2 - 2\cos\theta$

EXAMPLE 5.8 Arc Length of a Polar Curve

Find the arc length of the cardioid $r = 2 - 2\cos\theta$.

Solution A sketch of the cardioid is shown in Figure 9.49. First, notice that the curve is traced out with $0 \le \theta \le 2\pi$. From (5.8), the arc length is given by

$$\begin{aligned} s &= \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta = \int_0^{2\pi} \sqrt{(2\sin\theta)^2 + (2 - 2\cos\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\theta + 4 - 8\cos\theta + 4\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta = 16, \end{aligned}$$

where we leave the details of the integration as an exercise. (Hint: Use the half-angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to simplify the integrand. Be careful: remember that $\sqrt{x^2} = |x|$!) ■

EXERCISES 9.5

WRITING EXERCISES

1. Explain why the tangent line is perpendicular to the radius line at any point at which r is a local maximum. (See example 5.2.) In particular, if the tangent and radius are not perpendicular at (r, θ), explain why r is not a local maximum.
2. In example 5.5, explain why integrating from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ would give the area shown in Figure 9.46b and not the desired area.
3. Referring to example 5.6, explain why intersections can occur in each of the cases $f(\theta) = g(\theta)$, $f(\theta) = -g(\theta + \pi)$ and $f(\theta_1) = g(\theta_2) = 0$.
4. In example 5.7, explain why the length of the tank doesn't matter. If the problem were to compute the *amount* of oil left, would the length matter?

In exercises 1–10, find the slope of the tangent line to the polar curve at the given point.

1. $r = \sin 3\theta$ at $\theta = \frac{\pi}{3}$
2. $r = \sin 3\theta$ at $\theta = \frac{\pi}{2}$
3. $r = \cos 2\theta$ at $\theta = 0$
4. $r = \cos 2\theta$ at $\theta = \frac{\pi}{4}$
5. $r = 3\sin\theta$ at $\theta = 0$
6. $r = 3\sin\theta$ at $\theta = \frac{\pi}{2}$
7. $r = \sin 4\theta$ at $\theta = \frac{\pi}{4}$
8. $r = \sin 4\theta$ at $\theta = \frac{\pi}{16}$
9. $r = \cos 3\theta$ at $\theta = \frac{\pi}{6}$
10. $r = \cos 3\theta$ at $\theta = \frac{\pi}{3}$

In exercises 11–14, find all points at which $|r|$ is a maximum and show that the tangent line is perpendicular to the radius connecting the point to the origin.

11. $r = \sin 3\theta$
12. $r = \cos 4\theta$
13. $r = 2 - 4\sin 2\theta$
14. $r = 2 + 4\sin 2\theta$

In exercises 15–30, find the area of the indicated region.

15. One leaf of $r = \cos 3\theta$
16. One leaf of $r = \sin 4\theta$
17. Inner loop of $r = 3 - 4\sin\theta$
18. Inner loop of $r = 1 - 2\cos\theta$
19. Bounded by $r = 2\cos\theta$
20. Bounded by $r = 2 - 2\cos\theta$
21. Small loop of $r = 1 + 2\sin 2\theta$
22. Large loop of $r = 1 + 2\sin 2\theta$
23. Inner loop of $r = 2 + 3\sin 3\theta$
24. Outer loop of $r = 2 + 3\sin 3\theta$
25. Inside of $r = 3 + 2\sin\theta$ and outside of $r = 2$
26. Inside of $r = 2$ and outside of $r = 2 - 2\sin\theta$

27. Inside of $r = 2$ and outside of both loops of $r = 1 + 2\sin\theta$

28. Inside of $r = 2\sin 2\theta$ and outside $r = 1$

29. Inside of both $r = 1 + \cos\theta$ and $r = 1$

30. Inside of both $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

In exercises 31–34, find all points at which the two curves intersect.

31. $r = 1 - 2\sin\theta$ and $r = 2\cos\theta$

32. $r = 1 + 3\cos\theta$ and $r = -2 + 5\sin\theta$

33. $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

34. $r = 1 + \sqrt{3}\sin\theta$ and $r = 1 + \cos\theta$

In exercises 35–40, find the arc length of the given curve.

35. $r = 2 - 2\sin\theta$

36. $r = 3 - 3\cos\theta$

37. $r = \sin 3\theta$

38. $r = 2\cos 3\theta$

39. $r = 1 + 2\sin 2\theta$

40. $r = 2 + 3\sin 3\theta$

41. Repeat example 5.7 for the case where the oil stick shows a depth of 1.4.

42. Repeat example 5.7 for the case where the oil stick shows a depth of 1.0.

43. Repeat example 5.7 for the case where the oil stick shows a depth of 2.4.

44. Repeat example 5.7 for the case where the oil stick shows a depth of 2.6.

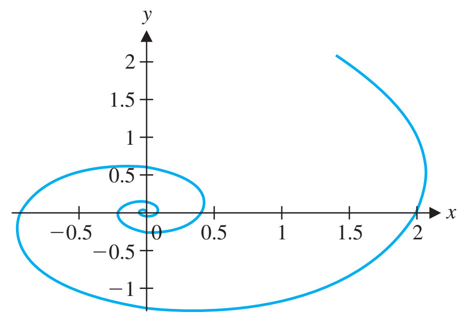
45. The problem of finding the slope of $r = \sin 3\theta$ at the point (0, 0) is not a well-defined problem. To see what we mean, show that the curve passes through the origin at $\theta = 0$, $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$, and find the slopes at these angles. Briefly explain why they are different even though the point is the same.

46. For each of the three slopes found in exercise 45, illustrate with a sketch of $r = \sin 3\theta$ for θ-values near the given values (e.g., $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$ to see the slope at $\theta = 0$).

47. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, has length L, show that $r = cf(\theta)$, $a \le \theta \le b$, has length $|c|L$ for any constant c.

48. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, encloses area A, show that for any constant c, $r = cf(\theta)$, $a \le \theta \le b$, encloses area $c^2 A$.

49. A logarithmic spiral is the graph of $r = ae^{b\theta}$ for positive constants a and b. The accompanying figure shows the case where $a = 2$ and $b = \frac{1}{4}$ with $\theta \le 1$. Although the graph never reaches the origin, the limit of the arc length from $\theta = d$ to a given point with $\theta = c$, as d decreases to $-\infty$, exists. Show that this total arc length equals $\frac{\sqrt{b^2+1}}{b}R$, where R is the distance from the starting point to the origin.



50. For the logarithmic spiral of exercise 49, if the starting point P is on the x-axis, show that the total arc length to the origin equals the distance from P to the y-axis along the tangent line to the curve at P.

EXPLORATORY EXERCISES

1. In this exercise, you will discover a remarkable property about the area underneath the graph of $y = \frac{1}{x}$. First, show that a polar representation of this curve is $r^2 = \frac{1}{\sin\theta\cos\theta}$. We will find the area bounded by $y = \frac{1}{x}$, $y = mx$ and $y = 2mx$ for $x > 0$, where m is a positive constant. Sketch graphs for $m = 1$ (the area bounded by $y = \frac{1}{x}$, $y = x$ and $y = 2x$) and $m = 2$ (the area bounded by $y = \frac{1}{x}$, $y = 2x$ and $y = 4x$). Which area looks larger? To find out, you should integrate. Explain why this would be a very difficult integration in rectangular coordinates. Then convert all curves to polar coordinates and compute the polar area. You should discover that the area equals $\frac{1}{2}\ln 2$ for any value of m. (Are you surprised?)

2. In the study of biological oscillations (e.g., the beating of heart cells), an important mathematical term is **limit cycle.** A simple example of a limit cycle is produced by the polar coordinates initial value problem $\frac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ and $\frac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$. Here, a is a positive constant. In section 7.2, we showed that the solution of the initial value problem $\frac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ is

$$r(t) = \frac{r_0}{r_0 - (r_0 - 1)e^{-at}}$$

and it is not hard to show that the solution of the initial value problem $\frac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$ is $\theta(t) = 2\pi t + \theta_0$. In rectangular coordinates, the solution of the combined initial value