DEFINITION. The IVP

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is a well-posed problem if:

(1) a unique solution $y(t)$ exists;

(2) There exists $\epsilon_0 > 0$ and $k > 0$ such that for any ϵ with $\epsilon_0 > \epsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \epsilon$ for all $t \in [a, b]$, and when $|\delta_0| < \epsilon$, the IVP

$$(\star) \quad \frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ such that

$$|z(t) - y(t)| < k\epsilon \text{ for all } t \in [a, b].$$

$(\star)$ is a perturbed problem (note that round-off errors cause perturbations).

THEOREM. *Given $D = \{(t, y) : a \le t \le b, -\infty \le y \le \infty\}$, if $f(t, y)$ is continuous and satisfies a Lipschitz condition on D in y, then the IVP*

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well-posed.

COROLLARY (Well-Posed Condition from text). *Let $D = \{(t, y) : a \le t \le b, -\infty \le y \le \infty\}$. Suppose that f and f_y, its first partial derivative with respect to y, are continuous for t in D. Then the IVP*

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

has a unique solution $y(t)$ for $a \le t \le b$, and the problem is well-posed.

EXAMPLE (continued). $f(t, y) = \frac{2}{t}y + t^2e^t$ is continuous on D and satisfies a Lipschitz condition on D in y, so the IVP is well-posed. □

So how do we approximate its unique solution?