Rates of convergence for sequences:

Our algorithms often generate sequences of values. Suppose $\{\alpha_n\} \to \alpha$ and $\beta_n \to 0$. Recall

$$\{\alpha_n\} = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$$

and

$$\{\alpha_n\} \to \alpha \text{ means } \lim_{n\to\infty} \alpha_n = \alpha.$$

If there exists $K > 0$ such that

$$|\alpha_n - \alpha| \le K|\beta_n| \text{ for large } n,$$

we say $\{\alpha_n\}$ converges to α with a rate of convergence $O(\beta_n)$ ("big oh of β_n") or $\{\alpha_n\} \to \alpha$ with rate of convergence $O(\beta_n)$. Usually, $\beta_n = \dfrac{1}{n^p}$ for some $p > 0$. We want the largest such value p so that $\alpha_n = \alpha + O\Big(\dfrac{1}{n^p}\Big)$.

Example. Consider $\left\{\dfrac{2n+3}{n+7}\right\} \to 2$ and $\left\{\dfrac{2n^3+3}{n^3+7}\right\} \to 2$.

n	$\frac{2n+3}{n+7}$	$\frac{2n^3+3}{n^3+7}$	$\frac{1}{n}$	$\frac{1}{n^3}$
1	.625000	.625000	1.000000	1.000000
2	.777778	1.266667	.500000	.125000
3	.900000	1.676471	.333333	.037037
4	1.000000	1.845070	.250000	0.15625
5	1.083333	1.916667	.200000	.0080000
6	1.153846	1.950673	.1666687	.004630

$$\left|\frac{2n+3}{n+7} - 2\right| = \left|\frac{2n+3-2n-14}{n+7}\right| = \left|\frac{-11}{n+7}\right| = \frac{11}{n+7} \le 11 \cdot \frac{1}{n} \quad (\beta_n = \frac{1}{n})$$

so $\dfrac{2n+3}{n+7} = 2 + O\Big(\dfrac{1}{n}\Big)$ and the rate of convergence is $O\Big(\dfrac{1}{n}\Big)$ since $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.