$$\left|\frac{2n^3+3}{n^3+7}-2\right| = \left|\frac{2n^3+3-2n^3-14}{n^3+7}\right| = \left|\frac{-11}{n^3+7}\right| = \frac{11}{n^3+7} \le 11\cdot\frac{1}{n^3} \quad (\beta_n = \frac{1}{n^3})$$

so $\dfrac{2n^3+3}{n^3+7} = 2+O\Big(\dfrac{1}{n^3}\Big)$ and rate of convergence is $O\Big(\dfrac{1}{n^3}\Big)$ since $\lim\limits_{n\to\infty}\dfrac{1}{n^3} = 0$.
□

Problem (p.28 #10b). Find the rate of convergence of $\Big\{\sin\dfrac{1}{n^2}\Big\}$.

We know $\lim\limits_{n\to\infty}\dfrac{1}{n^2} = 0$. From calculus, $\lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1 \Rightarrow \lim\limits_{k^2\to 0}\dfrac{\sin k^2}{k^2} = 1 \Rightarrow$

$\Big($letting $k = \dfrac{1}{n}\Big)$ $\lim\limits_{n\to\infty}\dfrac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = 1 < 2$. Then, for large $n$, $\sin\dfrac{1}{n^2} < 2\cdot\dfrac{1}{n^2}$.

Since $\lim\limits_{n\to\infty}\dfrac{1}{n^2} = 0$, $\sin\dfrac{1}{n^2} = 0 + O\Big(\dfrac{1}{n^2}\Big)$, so rate of convergence is $O\Big(\dfrac{1}{n^2}\Big)$. □

We can also use the big oh notation to show how some divergent sequences grow as $n$ becomes large. If positive constants $p$ and $K$ exist with

$$|\alpha_n| \ge Kn^p \text{ for all large values of } n,$$

we say that $\{\alpha_n\}\to\infty$ with rate of convergence $O(n^p)$.

Finally, we can use big oh for functions.

Suppose $\lim\limits_{h\to 0} G(h) = 0$ and $\lim\limits_{h\to 0} F(h) = L$. If there is $K > 0$ such that

$|F(h) - L| \le K|G(h)|$ for $h$ small enough, then $F(h) = L + O(G(h))$.

Usually, we like to use $G(h) = h^p$, and want the largest $p$ such that

$$F(h) = L + O(h^p).$$