EXAMPLE (Gaussian Elimination with Backward Substitution).

Notice how the row operations on the matrices mirror the corresponding operations on the equations. Notice also that the coding can apply to either the equations or the matrix.

$$\begin{cases} (1) & 4x_1 + x_2 + 2x_3 = 9 \\ (2) & 2x_1 + 4x_2 - x_3 = -5 \\ (3) & x_1 + x_2 - 3x_3 = -9 \end{cases} \qquad \left[\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right]$$

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$$\begin{cases} (1) & x_1 + x_2 - 3x_3 = -9 \quad (E_1) \leftrightarrow (E_3) \\ (2) & 2x_1 + 4x_2 - x_3 = -5 \\ (3) & 4x_1 + x_2 + 2x_3 = 9 \quad (E_1) \leftrightarrow (E_3) \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -3 & -9 \\ 2 & 4 & -1 & -5 \\ 4 & 1 & 2 & 9 \end{array}\right]$$

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$$\begin{cases} (1) & 2x_1 + 2x_2 - 6x_3 = -18 \quad (2E_1) \rightarrow (E_1) \\ (2) & 2x_1 + 4x_2 - x_3 = -5 \\ (3) & 4x_1 + x_2 + 2x_3 = 9 \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 2 & -6 & -18 \\ 2 & 4 & -1 & -5 \\ 4 & 1 & 2 & 9 \end{array}\right]$$

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$$\begin{cases} (1) & 2x_1 + 2x_2 - 6x_3 = -18 \\ (2) & 2x_2 + 5x_3 = 13 \quad (E_2 - 1E_1) \rightarrow (E_2) \\ (3) & -3x_2 + 14x_3 = 45 \quad (E_3 - 2E_1) \rightarrow (E_3) \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 2 & -6 & -18 \\ 0 & 2 & 5 & 13 \\ 0 & -3 & 14 & 45 \end{array}\right]$$

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$$\begin{cases} (1) & 2x_1 + 2x_2 - 6x_3 = -18 \\ (2) & 2x_2 + 5x_3 = 13 \\ (3) & \frac{43}{2}x_3 = \frac{129}{2} \quad (E_3 + \frac{3}{2}E_2) \rightarrow (E_3) \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 2 & -6 & -18 \\ 0 & 2 & 5 & 13 \\ 0 & 0 & \frac{43}{2} & \frac{129}{2} \end{array}\right]$$

This system of equations is (upper) triangular or reduced, and can be solved by backward substitution. The solution found thus applies to all five equivalent systems.