RECALL.

$$e^h = 1 + h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!} + \frac{e^\xi}{(n+1)!}h^{n+1}$$

$$\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots + \frac{(-1)^n h^{2n+1}}{(2n+1)!} + \frac{(-1)^{n+1}\cos\xi}{(2n+3)!}h^{2n+3}$$

$$\cos h = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots + \frac{(-1)^n h^{2n}}{(2n)!} + \frac{(-1)^{n+1}\cos\xi}{(2n+2)!}h^{2n+2}$$

$$\frac{1}{1+h} = 1 - h + h^2 - \cdots + (-1)^n h^n + \frac{(-1)^{n+1}}{(1+\xi)^{n+2}}h^{n+1}$$

PROBLEM (p.28 #11b). Find the rate of convergence for $\lim_{h\to 0}\frac{1-e^h}{h} = -1$.

$$e^h = 1 + h + \frac{e^\xi}{2}h^2 \implies 1 - e^h = -h - \frac{e^\xi}{2}h^2 \implies \frac{1-e^h}{h} = -1 - \frac{e^\xi}{2}h \implies$$

$\left|\frac{1-e^h}{h} + 1\right| = \left|-\frac{e^\xi}{2}h\right| < \frac{2}{2}|h|$ for h small enough. Thus $\frac{1-e^h}{h} = -1 + O(h)$.□

EXAMPLE. Find $\lim_{h\to 0}\frac{\cos h - 1 + \frac{1}{2}h^2}{h^4}$ and its rate of convergence.

$$\cos h = 1 - \frac{1}{2}h^2 + \frac{1}{24}h^4 - \frac{\cos\xi}{720}h^6 \text{ for } 0 < \xi < h \implies$$

$$\cos h - 1 + \frac{1}{2}h^2 = \frac{1}{24}h^4 - \frac{\cos\xi}{720}h^6 \implies$$

$$\frac{\cos h - 1 + \frac{1}{2}h^2}{h^4} = \underbrace{\frac{1}{24}}_{\text{Looks like } L} \underbrace{-\frac{\cos\xi}{720}h^2}_{+O(h^2)} \implies$$

$$\left|\frac{\cos h - 1 + \frac{1}{2}h^2}{h^4} - \frac{1}{24}\right| = \left|\frac{\cos\xi}{720}h^2\right| \le \frac{1}{720}|h^2|.$$