i.e., we need to choose constants $a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}$ such that

$$E(S_n) = \sum_{j=0}^{2n-1} \left\{ y_j - \left[\frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} \Big(a_k \cos kn + b_n \sin kx \Big) \right] \right\}^2 .$$

Finding the constants is simplified by the fact that $\{\phi_0, \phi_1, \ldots, \phi_{2n-1}\}$ is orthogonal with respect to summation over equally spaced points $\big\{(x_j, y_j)\big\}_{j=0}^{2m-1}$ in $[-\pi, \pi]$, i.e.,

$$\text{for } k \neq l, \quad \sum_{j=0}^{2m-1} \phi_k(x_j)\phi_l(x_j) = 0.$$

Why?

THEOREM. *If r is not a multiple of $2m$, r an integer,*

$$\sum_{j=0}^{2m-1} \cos r x_j = 0 \quad \textit{and} \quad \sum_{j=0}^{2m-1} \sin r x_j = 0,$$

and if r is not a multiple of m,

$$\sum_{j=0}^{2m-1} (\cos r x_j)^2 = m \quad \textit{and} \quad \sum_{j=0}^{2m-1} (\sin r x_j)^2 = m.$$

Proof.

RECALL. :

(1) $\displaystyle\sum_{i=1}^{n} r^i = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$.

(2) $i^2 = -1$.

(3) $e^{iz} = \cos z + i \sin z$.