6. Muller's Method

Roots of Polynomials

DEFINITION. A solution of $f(x) = 0$ is a zero of multiplicity m if for $x \neq p$,

$$f(x) = (x-p)^m q(x)$$

where

$$\lim_{x \to p} q(x) = 0.$$

A simple root is a root of mutiplicity one.

THEOREM. *$f \in C^1[a,b]$ has a simple zero at $p \in (a,b) \iff f(p) = 0$ but $f'(p) \neq 0$.*

THEOREM. *$f \in C^m[a,b]$ has a zero of multiplicity m at $p \in (a,b) \iff$*

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p)$$

but $f^{(m)}(p) \neq 0$.

NOTE. Zeros of multiplicity greater than 1 may not reach quadratic convergence with Newton's method.

A polynomial of degree n is of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

The polynomial $P(x) = 0$ has no degree.

THEOREM (Rational Root). *If $\frac{a}{b} \in \mathbb{Q}$ is a rational root of $P(x) = 0$ where $a, b \in \mathbb{Z}$, then a is a divisor of a_0 and b is a divisor of a_n.*

THEOREM (Fundamental Theorem of Algebra). *If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly) complex root.*