Example. Approximate $\sin(42^\circ)$ with error less than or equal to 10^{-6}.

$$x = 42^\circ = 45^\circ - 3^\circ = \frac{\pi}{4} - \frac{\pi}{60} = \frac{7\pi}{30}$$

Use Taylor polynomial with $x_0 = \dfrac{\pi}{4}$, so $x - x_0 = -\dfrac{\pi}{60}$.

To find n, with $f(x) = \sin(x)$ and $\dfrac{7\pi}{30} \le \xi(x) \le \dfrac{\pi}{4}$,

$$\left|R_n\Big(\frac{7\pi}{30}\Big)\right| = \frac{|f^{(n+1)}(\xi(x))|}{(n+1)!}\left|\frac{7\pi}{30} - \frac{\pi}{4}\right|^{n+1} \le \frac{1}{(n+1)!}\Big(\frac{\pi}{60}\Big)^{n+1} \approx \frac{.05236^{n+1}}{(n+1)!}.$$

Then $\left|R_2\Big(\dfrac{7\pi}{30}\Big)\right| \approx \dfrac{.05236^3}{3!} = .0000239$ and

$\left|R_3\Big(\dfrac{7\pi}{30}\Big)\right| \approx \dfrac{.05236^4}{4!} = .000000313 < 10^{-6}$, so take $n = 3$.

$$f(x) = \sin(x), \quad f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x),$$

so

$$f\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}, \quad f'\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}, \quad f''\Big(\frac{\pi}{4}\Big) = -\frac{\sqrt{2}}{2}, \quad f'''\Big(\frac{\pi}{4}\Big) = -\frac{\sqrt{2}}{2}.$$

Thus

$$\begin{aligned}\sin(42^\circ) &= \sin\Big(\frac{7\pi}{30}\Big) \approx P_3\Big(\frac{7\pi}{30}\Big)\\ &= f\Big(\frac{\pi}{4}\Big) + f'\Big(\frac{\pi}{4}\Big)(x - x_0) + \frac{f''(\frac{\pi}{4})}{2!}(x - x_0)^2 + \frac{f'''(\frac{\pi}{4})}{3!}(x - x_0)^3\\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\Big(-\frac{\pi}{60}\Big) - \frac{\sqrt{2}}{4}\Big(-\frac{\pi}{60}\Big)^2 - \frac{\sqrt{2}}{12}\Big(-\frac{\pi}{60}\Big)^3 = .6691304\end{aligned}$$

From the TI-89, $\sin\Big(\dfrac{7\pi}{30}\Big) = .6691306$,

so $\left|\sin(42^\circ) - P_3\Big(\dfrac{7\pi}{30}\Big)\right| = .0000002 = 2 \times 10^{-7}$. $\square$

Maple. See taylor's error.mw and/or taylor's error.pdf