CHAPTER 2

# The Real Numbers

## 2.1. The Algebraic and Order Properties of $\mathbb{R}$

Definition. A binary operation on a set $F$ is a function $B : F \times F \to F$.

For the binary operations of $+$ and $\cdot$, we replace $B(a, b)$ by $a + b$ and $a \cdot b$, respectively.

Field Axioms of $\mathbb{R}$

The real numbers are a field (as are the rational numbers $\mathbb{Q}$ and the complex numbers $\mathbb{C}$). That is, there are binary operations $+$ and $\cdot$ defined on $\mathbb{R} \ni$

(A1) $a + b = b + a \ \forall a, b \in \mathbb{R}$.

(A2) $(a + b) + c = a + (b + c) \ \forall a, b, c \in \mathbb{R}$.

(A3) $\exists\ 0 \in \mathbb{R} \ni 0 + a = a$ and $a + 0 = a \ \forall a \in \mathbb{R}$.

(A4) $\forall a \in \mathbb{R}, \exists\ -a \in \mathbb{R} \ni a + (-a) = 0$ and $(-a) + a = 0$.

(M1) $ab = ba \ \forall a, b \in \mathbb{R}$.

(M2) $(ab)c = a(bc) \ \forall a, b, c \in \mathbb{R}$.

(M3) $\exists\ 1 \in \mathbb{R}, 1 \neq 0, \ni 1 \cdot a = a$ and $a \cdot 1 = a \ \forall a \in \mathbb{R}$.

(M4) $\forall a \in \mathbb{R}, a \neq 0, \exists\ \dfrac{1}{a} \in \mathbb{R} \ni a\left(\dfrac{1}{a}\right) = 1$ and $\left(\dfrac{1}{a}\right)a = 1$.

(D) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca \ \forall a, b, c \in \mathbb{R}$.