Some Properties of $\mathbb{R}$

Theorem (2). *If $z, a \in \mathbb{R} \ni z + a = a$, then $z = 0$.*

(i.e., the number 0 guaranteed by (A3) is unique.)

Proof. By (A4), $\exists\ -a \in \mathbb{R} \ni a + (-a) = 0$.

Then

$$\begin{aligned} z & \\ &\underset{\text{A3}}{=} z + 0 = z + \big[a + (-a)\big] \underset{\text{A2}}{=} (z + a) + (-a) \\ &= a + (-a) = 0. \end{aligned}$$

□

Theorem (3). *Let $a, b \in \mathbb{R}$. Then $a + x = b$ has the unique solution $x = (-a) + b$.*

(i.e., we are difining $a - b$.)

Proof.

$$a + \big[(-a) + b\big] \underset{\text{A2}}{=} \big[a + (-a)\big] + b \underset{\text{A4}}{=} 0 + b \underset{\text{A3}}{=} b,$$

so $(-a) + b$ is a solution.

For uniqueness, suppose y is any solution of the equation, i.e., $a + y = b$. Then

$$\begin{aligned} y & \\ &\underset{\text{A3}}{=} 0 + y \\ &\underset{\text{A4}}{=} \big[(-a) + a\big] + y \\ &\underset{\text{A2}}{=} (-a) + (a + y) \\ &= (-a) + b \end{aligned}$$

□