Homework

Page 201 # 8

(E1) Let $f(x) = \begin{cases} 0, & x \in [0,2], x \neq 1 \\ 1, & x = 1 \end{cases}$. Show $\displaystyle\int_0^2 f = 0$.

(E2) Let $g(x) = \begin{cases} 0, & x \in [0,1] \setminus \mathbb{Q} \\ 1, & x \in [0,1] \cap \mathbb{Q} \end{cases}$. Show $g \notin \mathcal{R}[0,1]$.

7.2. Riemann Integrable Functions

THEOREM (7.2.1 — Cauchy Criterion). *$f : [a,b] \to \mathbb{R}$ is in $\mathcal{R}[a,b] \iff \forall\ \epsilon > 0\ \exists\ \eta_\epsilon > 0 \ni$ if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a,b]$ with $\|\dot{P}\| < \eta_\epsilon$ and $\|\dot{Q}\| < \eta_\epsilon$, then $\left|S(f;\dot{P}) - S(f;\dot{Q})\right| < \epsilon$.*

PROOF.

($\Longrightarrow$) Let $f \in \mathcal{R}[a,b]$ with $\displaystyle\int_a^b f = l$. Given $\epsilon > 0$,

$\exists\ \eta_\epsilon = \delta_{\epsilon/2} \ni$ if https://web1.ncaa.org/coachesTest/exec/proctorlogin are any tagged partitions of $[a,b]$ with $\|\dot{P}\| < \eta_\epsilon$ and $\|\dot{Q}\| < \eta_\epsilon$, then

$$\left|S(f;\dot{P}) - L\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|S(f;\dot{Q}) - L\right| < \frac{\epsilon}{2}.$$

Then

$$\left|S(f;\dot{P}) - S(f;\dot{Q})\right| = \left|S(f;\dot{P}) - L + L - S(f;\dot{Q})\right| \leq$$
$$\left|S(f;\dot{P}) - L\right| + \left|S(f;\dot{Q}) - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$