NOTE.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \underbrace{\mathbb{Q} \subseteq \mathbb{R} \subseteq C}_{\text{all fields}}$$

The last three all satisfy the field axioms, so the field axioms do not characterize $\mathbb{R}$. Recall that $\mathbb{Q}$ is closed under $+$ and $\cdot$, i.e., if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$ and $a \cdot b \in \mathbb{Q}$.

Homework Pages 29-30 #4 (Hint: assume $a \neq 0$ and prove $a = 1$), 5 (Hint: $1/(ab) = (1/a) \cdot (1/b)$ if $(1/a) \cdot (1/b)$ does what $1/(ab)$ is supposed to do), 8b (1st part)

Order Properties of $\mathbb{R}$

$\mathbb{R}$ is an ordered field, i.e., the following properties are satisfied:

(1) (Trichotomy) For $a, b \in \mathbb{R}$, exactly one of the following is true: $a < b$, $a = b$, or $a > b$.

(2) (Transitive) For $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(3) For $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(4) For $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, $ac < bc$.

Some Order Properties

THEOREM (4). *If $a, b \in \mathbb{R}$, then $a < b \iff -a > -b$.*

PROOF.

$$\begin{aligned}
&a < b \iff \\
&a + \big[(-a) + (-b)\big] < b + \big[(-a) + (-b)\big] \iff \\
&\big[a + (-a)\big] + (-b) < b + \big[(-b) + (-a)\big] \iff \\
&0 + (-b) < \big[b + (-b)\big] + (-a) \iff \\
&-b < 0 + (-a) \iff -b < -a \iff -a > -b
\end{aligned}$$

□