Theorem (5). *If $a, b, c \in \mathbb{R}$, then $a < b$ and $c < 0 \Longrightarrow ac > bc$*

Proof. $c < 0 \underset{\text{TH4}}{\Longrightarrow} -c > 0$.

Then $a(-c) < b(-c) \Longrightarrow -ac < -bc \underset{\text{TH4}}{\Longrightarrow} ac > bc$. □

Theorem (2.1.9). *If $a \in \mathbb{R} \ni 0 \leq a < \epsilon \ \forall \epsilon > 0$, then $a = 0$.*

Proof. Suppose $a > 0$.

Since $0 < \dfrac{1}{2} < 1$, $0 < \frac{1}{2}a < a$.

Let $\epsilon_0 = \frac{1}{2}a$.

Then $0 < \epsilon_0 < a$, contradicting our hypothesis. Thus $a = 0$. □

Problem (Page 30 #18). Let $a, b \in \mathbb{R}$, and suppose $a \leq b + \epsilon$ (or $a - \epsilon \leq b$) $\forall \epsilon > 0$. Then $a \leq b$.

Proof. By way of contradiction, suppose $b < a$.

[Need to find an ϵ that gives a contradiction.]

Let $\epsilon_0 = \dfrac{1}{2}(a - b)$. Then

$$a - \epsilon_0 = a - \frac{1}{2}(a - b) = \frac{1}{2}a + \frac{1}{2}b > \frac{1}{2}b + \frac{1}{2}b = b,$$

contradicting our hypothesis. Thus $a \leq b$. □