

EXAMPLE.

(7) The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The proof is similar to that of Example 5.

(8) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

PROOF. Since $(s - k)$ is monotone, we need only to show (s_k) is bounded. But it suffices to show that some subsequence is bounded.

Let $k_1 = 2^1 - 1 = 1$, so $s_{k_1} = 1$.

Let $k_2 = 2^2 - 1 = 3$. Then, since $2^p < 3^p$,

$$s_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}.$$

Let $k_3 = 2^3 - 1 = 7 \implies$

$$\begin{aligned} s_{k_3} &= s_{k_2} + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) \\ &< s_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}. \end{aligned}$$

Continuing inductively, if $k_j = 2^j - 1$,

$$\begin{aligned} 0 < s_{k_j} &= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{j-1})^{p-1}} \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots + \frac{1}{(2^{p-1})^{j-1}} \\ &< \frac{1}{1 - \frac{1}{2^{p-1}}}. \end{aligned}$$

Thus (s_{k_j}) is bounded and so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. □