THEOREM (Arithmetic-Geometric Mean Inequality).

Suppose $a, b > 0$. *Then*

$$\sqrt{ab} \le \frac{1}{2}(a+b)$$

with equality holding $\iff$ $a = b$.

PROOF.

(1) Suppose $a \ne b$. Then $\sqrt{a} > 0$, $\sqrt{b} > 0$, and $\sqrt{a} \ne \sqrt{b}$.

Thus

$$\sqrt{a} - \sqrt{b} \ne 0 \Longrightarrow$$
$$\left(\sqrt{a} - \sqrt{b}\right)^2 > 0 \Longrightarrow$$
$$a - 2\sqrt{a}\sqrt{b} + b > 0 \Longrightarrow$$
$$-2\sqrt{a}\sqrt{b} > -(a+b) \Longrightarrow$$
$$\sqrt{ab} < \frac{1}{2}(a+b)$$

(2) If $a = b$,

$$\sqrt{ab} = \sqrt{a^2} = |a| = a = \frac{1}{2}(2a) = \frac{1}{2}(a+a) = \frac{1}{2}(a+b).$$

(3) If $\sqrt{ab} = \frac{1}{2}(a+b)$,

$$ab = \frac{1}{4}\left(a+b\right)^2 \Longrightarrow$$
$$4ab = a^2 + 2ab + b^2 \Longrightarrow$$
$$0 = a^2 - 2ab + b^2 \Longrightarrow$$
$$0 = (a-b)^2 \Longrightarrow$$
$$0 = a - b \Longrightarrow$$
$$a = b.$$

$\square$