THEOREM (9.4.2 — Ratio Test). *Let (x_n) be a sequence of nonzero real numbers.*

(a) If $\exists\, r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N} \ni \left|\dfrac{x_{n+1}}{x_n}\right| \le r$ for $n \ge K$,

then $\sum x_n$ is absolutely convergent.

(b) If $\exists\, K \in \mathbb{N} \ni \left|\dfrac{x_{n+1}}{x_n}\right| \ge 1$ for $n \ge K$,

then $\sum x_n$ diverges.

PROOF.

(a) If $\left|\dfrac{x_{n+1}}{x_n}\right| \le r \ \forall\, n \ge K$, by induction

$$|x_{K+m}| \le |x_K| r^m \quad \text{for } m \in \mathbb{N}.$$

Thus, for $n \ge K$, the terms of $\sum |x_n|$ are dominated by the terms of $|x_K| \sum r^n$, so $\sum |x_n|$ converges by comparison.

(b) If $\left|\dfrac{x_{n+1}}{x_n}\right| \ge 1$ for $n \ge K$, by induction

$$|x_{K+m}| \ge |x_K| \quad \text{for } m \in \mathbb{N}$$

and so $\sum x_n$ diverges by the n'th-term test. □

COROLLARY (9.2.5). *Let (x_n) be a nonzero sequence in $\mathbb{R}$ and suppose $r = \lim \left|\dfrac{x_{n+1}}{x_n}\right|$ exists in $\mathbb{R}$. Then $\sum x_n$ converges absolutely for $r < 1$ and diverges for $r > 1$.*

EXAMPLE.

(4) For any p-series, absolutely convergent or divergent,

$$r = \lim \left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \lim \left(\frac{n+1}{n} \right)^p = 1^p = 1.$$