(4) $\lim \left(\dfrac{1}{2^n}\right) = 0.$

PROOF. Given $\epsilon > 0$.

$$\left|\frac{1}{2^n} - 0\right| < \epsilon \iff \frac{1}{2^n} < \epsilon \iff \frac{1}{\epsilon} < 2^n \iff$$

$$\ln \frac{1}{\epsilon} < \ln 2^n \iff -\ln \epsilon < n \ln 2 \iff \frac{-\ln \epsilon}{\ln 2} < n.$$

Take $K = \max\left\{1, \left[\dfrac{-\ln \epsilon}{\ln 2}\right] + 1\right\}$.

Then $n \geq K \implies n > \dfrac{-\ln \epsilon}{\ln 2} \implies \left|\dfrac{1}{2^n} - 0\right| < \epsilon.$ □

(5) Let $x_n = 1 + (-1)^n$. $X = (0, 2, 0, 2, \dots)$.

$\lim(x_n)$ does not exist.

PROOF. [We use contradiction.]

Suppose $\lim(x_n) = x$. Then, $\forall\ \epsilon > 0, \exists\ K \in \mathbb{N} \ni \forall\ n \geq K, |x_n - x| < \epsilon$.

In particular, for $\epsilon = 1$, $\exists\ K \in \mathbb{N} \ni \forall\ n \geq K, |x_n - x| < 1$.

But $\begin{cases} |0 - x| < 1 & \text{for } n \text{ odd} \\ |2 - x| < 1 & \text{for } n \text{ even} \end{cases}$,

so $2 = |2 - x + x| \leq |2 - x| + |x| \leq |2 - x| + |x - 0| < 1 + 1 = 2$,

a contradiction.

Thus $\lim(x_n)$ does not exist. □