THEOREM (3.2.7 — Squeeze Theorem). *Suppose* $x_n \le y_n \le z_n\ \forall n \in \mathbb{N}$ *and* $\lim(x_n) = \lim(z_n)$. *Then* (y_n) *converges and*

$$\lim(x_n) \le \lim(y_n) \le \lim(z_n).$$

PROOF. Let $w = \lim(x_n) = \lim(z_n)$. Given $\epsilon > 0$.

$\exists\ K_1 \in \mathbb{N} \ni \forall\ n \ge K_1,\ -\epsilon < x_n - w < \epsilon$, and also

$\exists\ K_2 \in \mathbb{N} \ni \forall\ n \ge K_2,\ -\epsilon < z_n - w < \epsilon$.

Let $K = \max\{K_1, K_2\}$. Then for $n \ge K$,

$$-\epsilon \underbrace{<}_{n \ge K_1} x_n - w \le y_n - w \le z_n - w \underbrace{<}_{n \ge K_2} \epsilon \Longrightarrow |y_n - w| < \epsilon.$$

thus $\lim(y_n) = w$. □

NOTE. The hypotheses of Theorem 3.2.4 thru Theorem 3.2.7 can be weakened to apply to tails of the sequences rather than to the sequences themselves.

EXAMPLE.

(1) Find $\lim\left(\dfrac{\cos n}{n}\right)$.

SOLUTION. $-1 \le \cos n \le 1 \Longrightarrow -\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$.

Since $\lim\left(-\dfrac{1}{n}\right) = \lim\left(\dfrac{1}{n}\right) = 0$,

$\lim\left(\dfrac{\cos n}{n}\right) = 0$ by the Squeeze Theorem. □