PROBLEM (Page 16 #10). Find and prove a formula for

$$P(n) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

SOLUTION. $P(1) = \frac{1}{3}$, $P(2) = \frac{1}{3} + \frac{1}{15} = \frac{6}{15} = \frac{2}{5}$,

$P(3) = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} = \frac{35+7+3}{105} = \frac{45}{105} = \frac{3}{7}$

Conjecture: $P(n) = \frac{n}{2n+1}$.

PROOF. Let $S \subseteq \mathbb{N} \ni P(n) = \frac{n}{2n+1}$.

$1 \in S$ since $P(1) = \frac{1}{1\cdot 3} = \frac{1}{3} = \frac{1}{2\cdot 1+1}$.

Suppose $k \in S$, i.e.,

$$P(k) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}.$$

Then

$$P(k+1) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{\big[(2(k+1)-1\big]\big[(2(k+1)+1\big]} =$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} =$$

$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} =$$

$$\frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{2(k+1)+1},$$

so $k+1 \in S$.

Thus, by math induction, $S = \mathbb{N}$. □

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