THEOREM (3.4.2). *If $X = (x_n)$ converges to x, so does any subsequence (x_{n_k}).*

PROOF. Let $\epsilon > 0$ be given. Since (x_n) converges to x,

$\exists\ K \in \mathbb{N} \ni \forall\ n \geq K,\ |x_n - x| < \epsilon$. Since

$$n_1 < n_2 < \cdots < n_k < \cdots$$

is an increasing sequence in $\mathbb{N}$, $n_k \geq k\ \forall\ k \in \mathbb{N}$.

Let $K' = n_K$. Then, $\forall\ n_k \geq K' = n_K,\ n_k \geq K \Longrightarrow |x_{n_k} - x| < \epsilon$.

Thus (x_{n_k}) converges to x. □

EXAMPLE. For $c > 1$, find $\lim(c^{\frac{1}{n}})$ if it exists.

SOLUTION.

(a) $x_n = c^{\frac{1}{n}} > 1\ \forall\ n \in \mathbb{N}$, so (x_n) is bounded below.

(b) $x_n - x_{n+1} = c^{\frac{1}{n}} - c^{\frac{1}{n+1}} = c^{\frac{1}{n+1}}\Big(c^{\frac{1}{n(n+1)}} - 1\Big) > 0\ \forall\ n \in \mathbb{N}$,

so (x_n) is decreasing.

(c) Thus $\lim(x_n) = x$ exists.

[Using a subsequence to find x.]

Now $x_{2n} = c^{\frac{1}{2n}} = \Big(c^{\frac{1}{n}}\Big)^{\frac{1}{2}} = \Big(x_n\Big)^{\frac{1}{2}}$, so

$$x = \lim(x_{2n}) = \lim\left(\Big(x_n\Big)^{\frac{1}{2}}\right) = x^{\frac{1}{2}} \Longrightarrow$$

$$x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0 \text{ or } x = 1.$$

Since $x_n > 1\ \forall\ n \in \mathbb{N}$, $\lim(x_n) = 1$. □