Example. $x_1 = 1,\ x_2 = 2,\ x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n \geq 3$.

$$(x_n) = \left(1, 2, \frac{3}{2}, \frac{7}{4}, \frac{13}{8}, \frac{27}{16}, \dots\right).$$

(a) (x_n) is contractive. Thus (x_n) converges.

Proof.

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{2}(x_n + x_{n+1}) - x_{n+1}\right| = \left|\frac{1}{2}x_n - \frac{1}{2}x_{n+1}\right| = \frac{1}{2}|x_{n+1} - x_n|.$$

□

(b) Note that

$$|x_{n+1} - x_n| = \frac{1}{2^{n-1}}|x_2 - x_1| = \frac{1}{2^{n-1}}.$$

(c) [To find $\lim(x_{2n+1}) = \lim(x_n)$.]

$$x_{2n+1} - x_{2n-1} = \frac{1}{2}(x_{2n-1} + x_{2n}) - x_{2n-1} = \frac{1}{2}x_{2n} - \frac{1}{2}x_{2n-1} =$$

$$\frac{1}{2}|x_{2n} - x_{2n-1}| = \frac{1}{2} \cdot \frac{1}{2^{2n-2}} = \frac{1}{2^{2n-1}}.$$

Thus

$$x_{2n+1} = x_{2n-1} + \frac{1}{2^{2n-1}} = x_{2n-3} + \frac{1}{2^{2n-3}} + \frac{1}{2^{2n-1}} = \cdots =$$

$$1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}} = 1 + \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2n-2}}\right] =$$

$$1 + \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2n-2}}\right] = 1 + \frac{1}{2}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{n-1}\right] =$$

$$1 + \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}} = 1 + \frac{1}{2} \cdot \frac{4 - \frac{1}{4^{n-1}}}{4 - 1} =$$

$$1 + \frac{4 - \frac{1}{4^{n-1}}}{6} \to 1 + \frac{2}{3} = \frac{5}{3} \text{ as } n \to \infty.$$

Thus $\lim(x_n) = \lim(x_{2n+1}) = \frac{5}{3}$.