EXAMPLE. $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist.

PROOF. Recall $\cos\theta = 1$ for $\theta = 2n\pi$ and $\cos\theta = -1$ for $\theta = (2n+1)\pi$. Let $(x_n) = \left(\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots\right) = \left(\frac{1}{n\pi}\right)$. $(x_n) \to 0$, but

$$\left(\cos\frac{1}{x_n}\right) = \big(\cos\pi, \cos 2\pi, \dots\big) = (-1, 1, -1, 1, \dots) \text{ diverges.}$$

Thus $\cos\left(\frac{1}{x}\right)$ diverges.

Another view:

$\left(\frac{1}{(2n-1)\pi}\right) \to 0$ and

$$\left(\cos\left(\frac{1}{\frac{1}{(2n-1)\pi}}\right)\right) = \big(\cos(2n-1)\pi\big) = (-1, -1, -1, \dots) \to -1, \text{ while}$$

$$\left(\cos\left(\frac{1}{\frac{1}{2n\pi}}\right)\right) = \big(\cos 2n\pi\big) = (1, 1, 1, \dots) \to 1,$$

i.e., we found two sequences converging to 0 that gave different limits. □

$\boxed{\textit{Homework}}$ Page 104 # 9bc, 10ab, 11d, 14