

EXAMPLE.

(3) $f(x) = x^{73}$ is u.c. on $[-13, 13]$.

(4) $f(x) = \sqrt{x}$ is u.c. on $[0, 400]$.

(5) $f(x) = x^{17} \sin e^x - e^{4x} \cos(2x)$ is u.c. on $[-8\pi, 8\pi]$.

(6) $f(x) = \frac{1}{x^6}$ is u.c. on $\left[\frac{1}{4}, 44\right]$.

What if f is continuous on a set that is not a closed, bounded interval?

DEFINITION (5.4.4). Let $A \subseteq \mathbb{R}$ and $F : A \rightarrow \mathbb{R}$. If

$$\exists K > 0 \ni |f(x) - f(u)| \leq K|x - u| \quad \forall x, u \in A,$$

then f is a Lipschitz function (or satisfies a Lipschitz condition) on A .

NOTE. This is equivalent to

$$\left| \frac{f(x) - f(u)}{x - u} \right| \leq K$$

on I , i.e., the slopes with all line segments with endpoints in I is bounded by K .

THEOREM (5.4.5). *If $f : A \rightarrow \mathbb{R}$ is a Lipschitz function on A , then f is uniformly continuous on A .*

PROOF. Given $\epsilon > 0$. Suppose f is Lipschitz with constant K .

Take $\delta = \frac{\epsilon}{K}$. Then

$$x, u \in A \text{ and } |x - u| < \delta \implies |f(x) - f(u)| \leq K|x - u| < K \cdot \frac{\epsilon}{K} = \epsilon.$$

Thus f is u.c. on A . □